An Extension of Complex Role Inclusion Axioms in the Description Logic $\mathcal{SROIQ}$

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Outline

Description Logics

- A family of knowledge representation languages

$$\text{Myocardium} \equiv \text{Muscle} \sqcap \exists \text{isPartOf}.\text{Heart}$$

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- The basic DL $\mathcal{ALC}$ [Schmidt-Schauß, Smolka; 1991]:

Name	DL syntax	First-Order syntax
conjunction	$C_1 \sqcap C_2$	$C_1(x) \wedge C_2(x)$
disjunction	$C_1 \sqcup C_2$	$C_1(x) \vee C_2(x)$
negation	$\neg C$	$\neg C(x)$
existential restriction	$\exists r.C$	$\exists y.[r(x, y) \wedge C(y)]$
value restriction	$\forall r.C$	$\forall y.[r(x, y) \rightarrow C(y)]$

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  - is a subset of $\mathcal{GF}^2$

Complex Role Inclusion Axioms

- $\mathcal{SROIQ}$ [Horrocks,Kutz,Sattler;2006]
- A very expressive DL
- The basis of W3C ontology web language OWL 2
- One of the powerful features of $\mathcal{SROIQ}$ are:

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complex RIA	$R_1 \cdot R_2 \sqsubseteq R_3$	$\forall xyz.[R_1(x,y) \wedge R_2(y,z) \rightarrow R_3(x,z)]$

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- Closely related to:
  - Grammar logics [Fariñas del Cerro, Penttonen; 1998], [Baldoni;1998], [Demri; 2001]
  - First-order theories with compositional binary relations [Bachmair, Ganzinger; 1998]

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Example

$L(R) = \{{S_1}^* \cdot R \qquad \}$

$S_1 \cdot R \sqsubseteq R$

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Example

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$$R \cdot S_2 \sqsubseteq R$$

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Example

$L(R) = \{{S_1}^* \cdot R \cdot {S_2}^*\}$
$S_1 \cdot R \sqsubseteq R$
$R \cdot S_2 \sqsubseteq R$
– Regular

$L(R) = \{{S_1}^n \cdot R \cdot {S_2}^n\}$
$S_1 \cdot R \cdot S_2 \sqsubseteq R$
– Not regular

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EXAMPLE

$L(R) = \{S_1{}^* \cdot R \cdot S_2{}^*\}$	$L(R) = \{S_1{}^n \cdot R \cdot S_2{}^n\}$
$S_1 \cdot R \sqsubseteq R$	$S_1 \cdot R \cdot S_2 \sqsubseteq R$
$R \cdot S_2 \sqsubseteq R$	
– Regular	– Not regular

- How to ensure that the set of RIAs is regular?
  - Checking if a CFG is regular is undecidable

$\prec$-REGULARITY

- Sufficient condition for regularity:

$\prec$-REGULARITY

1. $R \cdot R \sqsubseteq R$ (transitivity)
2. $R^{-} \sqsubseteq R$ (symmetry)
3. $S_1 \cdots S_n \sqsubseteq R$
4. $R \cdot S_1 \cdots S_n \sqsubseteq R$ (left-linear)
5. $S_1 \cdots S_n \cdot R \sqsubseteq R$ (right-linear)

where $S_i \prec R$

- Where $\prec$ is an admissible order on roles:
  - $\prec$ is irreflexive
  - $\prec$ is transitive
  - $R_1 \prec R_2$ iff $R_1 \prec {R_2}^{-}$

Regularity vs. $\prec$-Regularity

Example

$\text{isPartOf} \cdot \text{isPartOf} \sqsubseteq \text{isPartOf} \rightsquigarrow$ **1**

$\prec$-Regularity

1. $R \cdot R \sqsubseteq R$
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Regularity vs. $\prec$-Regularity

Example

isPartOf · isPartOf ⊑ isPartOf ⇝ 1

isProperPartOf ⊑ isPartOf ⇝ 3

- isProperPartOf $\prec$ isPartOf

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isProperPartOf · isPartOf ⊑ isPartOf ⇝ 5

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$\text{isPartOf} \cdot \text{isProperPartOf} \sqsubseteq \text{isPartOf} \rightsquigarrow$ **4**

$\text{isProperPartOf} \cdot \text{isPartOf} \sqsubseteq \text{isPartOf} \rightsquigarrow$ **5**

- $\text{isProperPartOf} \prec \text{isPartOf}$

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Regularity vs. ≺-Regularity

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- isProperPartOf $\prec$ isPartOf
- isPartOf $\prec$ isProperPartOf
- There is no admissible $\prec$ such that set of RIAs is $\prec$-regular!

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- isProperPartOf $\prec$ isPartOf
- isPartOf $\prec$ isProperPartOf
- There is no admissible $\prec$ such that set of RIAs is $\prec$-regular!
- However the set of RIAs is regular:

$$L(\text{isPartOf}) = (\text{isPartOf} \mid \text{isProperPartOf})^+$$
$$L(\text{isProperPartOf}) = L(\text{isPartOf}) \setminus \text{isPartOf}^*$$

USE CASES

1. SEP Triplets encoding (used, e.g., in SNOMED CT):
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$$\text{NonPartitivelyContaines} \sqsubseteq \text{Contains}$$

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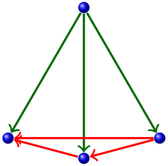

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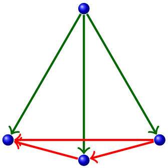

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1. They guarantee regularity for the set of RIAs
2. Corresponding NFAs can be effectively constructed
3. Can be checked in polynomial time
4. They are backward compatible with the original restrictions
5. For every regular set of RIAs there exists a conservative extension that satisfies our restrictions.

Outline

1 Introduction

2 The new restrictions on RIAs

Associativity

Example

$$\text{isPartOf} \cdot \text{isPartOf} \sqsubseteq \text{isPartOf}$$
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$$\text{isPartOf} \cdot \text{isProperPartOf} \cdot \text{isPartOf} \cdot \text{isPartOf} \sqsubseteq \text{isProperPartOf}$$

- It does not matter in which order the RIAs are applied!

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A set of RIAs is right associative if:

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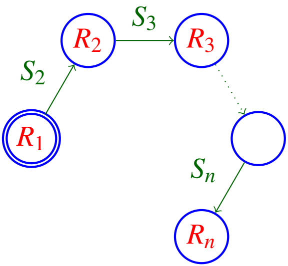

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A set of RIAs is left associative if $\rho_1(R\rho_2) \sqsubseteq^* R' \Rightarrow (\rho_1 R)\rho_2 \sqsubseteq^* R'$

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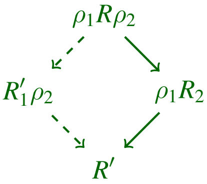
  - the proof is analogous to the Church-Rosser property:



Why Associativity is Better?

- There are regular but not associative sets of RIAs:

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Theorem

Every regular set of RIAs can be conservatively extended to an associative set of RIAs.

Proof.

Not constructive. Application of the Myhill-Nerode theorem. □

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   Regularity of RIAs can be reduced to regularity of CFGs:

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