An Extension of Regularity Conditions for Complex Role Inclusion Axioms

Yevgeny Kazakov

Oxford University Computing Laboratory

July 30, 2009

Outline

Complex Role Inclusion Axioms

- A new powerful feature in $\mathcal{SROIQ}$ and $\mathcal{OWL}$ 2
- Axioms of the form: $R_1 \cdots R_n \sqsubseteq R$

Example

hasParent · hasBrother $\sqsubseteq$ hasUncle

COMPLEX ROLE INCLUSION AXIOMS

- A new powerful feature in $\mathcal{SROIQ}$ and $\mathcal{OWL}$ **2**
- Axioms of the form: $R_1 \cdots R_n \sqsubseteq R$

EXAMPLE

hasParent · hasBrother $\sqsubseteq$ hasUncle

- In general cause undecidability (except for $\mathcal{EL}^{++}$)

Complex Role Inclusion Axioms

- A new powerful feature in $\mathcal{SROIQ}$ and $\mathcal{OWL}$ 2
- Axioms of the form: $R_1 \cdots R_n \sqsubseteq R$

Example

hasParent · hasBrother $\sqsubseteq$ hasUncle

- In general cause undecidability (except for $\mathcal{EL}^{++}$)
- Decidability is regained by imposing restrictions:

$\prec$-Regularity

1. $R \cdot R \sqsubseteq R$ (transitivity)
2. $R^- \sqsubseteq R$ (symmetry)
3. $S_1 \cdots S_n \sqsubseteq R$
4. $R \cdot S_1 \cdots S_n \sqsubseteq R$ (left-linear)
5. $S_1 \cdots S_n \cdot R \sqsubseteq R$ (right-linear)

where $S_i \prec R$

REGULARITY

- Complex RIAs are related to context-free grammars:

$$R \longrightarrow R_1 \cdots R_n \quad \leftrightsquigarrow \quad R_1 \cdots R_n \sqsubseteq R$$

Regularity

- Complex RIAs are related to context-free grammars:

$$R \longrightarrow R_1 \cdots R_n \quad \leftrightsquigarrow \quad R_1 \cdots R_n \sqsubseteq R$$

- Induce a language for every role R:

$$L(R) = \{R_1 \cdots R_n \mid R \twoheadrightarrow R_1 \ldots R_n\}$$

REGULARITY

- Complex RIAs are related to context-free grammars:

$$R \longrightarrow R_1 \cdots R_n \quad \leftrightsquigarrow \quad R_1 \cdots R_n \sqsubseteq R$$

- Induce a language for every role R:

$$L(R) = \{R_1 \cdots R_n \mid R \twoheadrightarrow R_1 \ldots R_n\}$$

or $$L(R) = \{R_1 \cdots R_n \mid R_1 \ldots R_n \sqsubseteq^* R\}$$

Regularity

- Complex RIAs are related to context-free grammars:

$$R \rightarrow R_1 \cdots R_n \quad \leftrightsquigarrow \quad R_1 \cdots R_n \sqsubseteq R$$

- Induce a language for every role R:

$$L(R) = \{R_1 \cdots R_n \mid R \twoheadrightarrow R_1 \ldots R_n\}$$

or $$L(R) = \{R_1 \cdots R_n \mid R_1 \ldots R_n \sqsubseteq^* R\}$$

- $\prec$-regularity implies that $L(R)$ is regular for every R

Regularity

- Complex RIAs are related to context-free grammars:

$$R \longrightarrow R_1 \cdots R_n \quad \leftrightsquigarrow \quad R_1 \cdots R_n \sqsubseteq R$$

- Induce a language for every role R:

$$L(R) = \{R_1 \cdots R_n \mid R \twoheadrightarrow R_1 \ldots R_n\}$$

or $$L(R) = \{R_1 \cdots R_n \mid R_1 \ldots R_n \sqsubseteq^* R\}$$

- $\prec$-regularity implies that $L(R)$ is regular for every R

Example

$$L(R) = \{ \quad R \quad \}$$

Regularity

- Complex RIAs are related to context-free grammars:

$$R \longrightarrow R_1 \cdots R_n \quad \leftrightsquigarrow \quad R_1 \cdots R_n \sqsubseteq R$$

- Induce a language for every role R:

$$L(R) = \{R_1 \cdots R_n \mid R \twoheadrightarrow R_1 \ldots R_n\}$$

or $$L(R) = \{R_1 \cdots R_n \mid R_1 \ldots R_n \sqsubseteq^* R\}$$

- $\prec$-regularity implies that $L(R)$ is regular for every R

Example

$R \cdot R \sqsubseteq R$ $\qquad L(R) = \{ \quad R^+ \quad \}$

Regularity

- Complex RIAs are related to context-free grammars:

$$R \longrightarrow R_1 \cdots R_n \quad \leftrightsquigarrow \quad R_1 \cdots R_n \sqsubseteq R$$

- Induce a language for every role R:

$$L(R) = \{R_1 \cdots R_n \mid R \twoheadrightarrow R_1 \ldots R_n\}$$

or $$L(R) = \{R_1 \cdots R_n \mid R_1 \ldots R_n \sqsubseteq^* R\}$$

- $\prec$-regularity implies that $L(R)$ is regular for every R

Example

$R \cdot R \sqsubseteq R$ $\qquad L(R) = \{{S_1}^* \cdot R^+ \quad \}$

$S_1 \cdot R \sqsubseteq R$

Regularity

- Complex RIAs are related to context-free grammars:

$$R \longrightarrow R_1 \cdots R_n \quad \leftrightsquigarrow \quad R_1 \cdots R_n \sqsubseteq R$$

- Induce a language for every role R:

$$L(R) = \{R_1 \cdots R_n \mid R \twoheadrightarrow R_1 \dots R_n\}$$

or $$L(R) = \{R_1 \cdots R_n \mid R_1 \dots R_n \sqsubseteq^* R\}$$

- $\prec$-regularity implies that $L(R)$ is regular for every R

Example

$R \cdot R \sqsubseteq R$
$S_1 \cdot R \sqsubseteq R$
$R \cdot S_2 \sqsubseteq R$

$$L(R) = \{{S_1}^* \cdot R^+ \cdot {S_2}^*\}$$

REGULARITY

- Complex RIAs are related to context-free grammars:

$$R \longrightarrow R_1 \cdots R_n \quad \leftrightsquigarrow \quad R_1 \cdots R_n \sqsubseteq R$$

- Induce a language for every role R:

$$L(R) = \{R_1 \cdots R_n \mid R \twoheadrightarrow R_1 \ldots R_n\}$$

or $$L(R) = \{R_1 \cdots R_n \mid R_1 \ldots R_n \sqsubseteq^* R\}$$

- $\prec$-regularity implies that $L(R)$ is regular for every R

EXAMPLE

$R \cdot R \sqsubseteq R$ $\qquad L(R) = \{{S_1}^* \cdot R^+ \cdot {S_2}^*\}$

$S_1 \cdot R \sqsubseteq R$

$R \cdot S_2 \sqsubseteq R$

- The procedure for $\mathcal{SROIQ}$ requires just a NFA for $L(R)$
  $\Rightarrow$ works for any RIAs which induce regular languages

$\prec$-Regularity vs. Regularity

Example (hasProperPart $\prec$ hasPart)

$$\mathsf{hasProperPart} \sqsubseteq \mathsf{hasPart} \rightsquigarrow \boxed{3}$$
$$\mathsf{hasPart} \cdot \mathsf{hasPart} \sqsubseteq \mathsf{hasPart} \rightsquigarrow \boxed{1}$$

$\prec$-Regularity

1. $R \cdot R \sqsubseteq R$
2. $R^{-} \sqsubseteq R$
3. $S_1 \cdots S_n \sqsubseteq R$
4. $R \cdot S_1 \cdots S_n \sqsubseteq R$
5. $S_1 \cdots S_n \cdot R \sqsubseteq R$

$\prec$-Regularity vs. Regularity

Example (hasProperPart $\prec$ hasPart)

$$\text{hasProperPart} \sqsubseteq \text{hasPart} \rightsquigarrow 3$$
$$\text{hasPart} \cdot \text{hasPart} \sqsubseteq \text{hasPart} \rightsquigarrow 1$$

$$\text{hasPart} \cdot \text{hasProperPart} \sqsubseteq \text{hasPart} \rightsquigarrow 4$$
$$\text{hasProperPart} \cdot \text{hasPart} \sqsubseteq \text{hasPart} \rightsquigarrow 5$$

$\prec$-Regularity

1. $R \cdot R \sqsubseteq R$
2. $R^- \sqsubseteq R$
3. $S_1 \cdots S_n \sqsubseteq R$
4. $R \cdot S_1 \cdots S_n \sqsubseteq R$
5. $S_1 \cdots S_n \cdot R \sqsubseteq R$

$\prec$-Regularity vs. Regularity

Example (hasProperPart $\prec$ hasPart)

$$\text{hasProperPart} \sqsubseteq \text{hasPart} \rightsquigarrow \mathbf{3}$$
$$\text{hasPart} \cdot \text{hasPart} \sqsubseteq \text{hasPart} \rightsquigarrow \mathbf{1}$$

$$\text{hasPart} \cdot \text{hasProperPart} \sqsubseteq \underline{\text{hasPart}} \rightsquigarrow \mathbf{4}$$
$$\text{hasProperPart} \cdot \text{hasPart} \sqsubseteq \underline{\text{hasPart}} \rightsquigarrow \mathbf{5}$$

$\prec$-Regularity

1. $R \cdot R \sqsubseteq R$
2. $R^- \sqsubseteq R$
3. $S_1 \cdots S_n \sqsubseteq R$
4. $R \cdot S_1 \cdots S_n \sqsubseteq R$
5. $S_1 \cdots S_n \cdot R \sqsubseteq R$

$\prec$-Regularity vs. Regularity

Example (hasProperPart $\prec$ hasPart)

$$\text{hasProperPart} \sqsubseteq \text{hasPart} \rightsquigarrow \boxed{3}$$
$$\text{hasPart} \cdot \text{hasPart} \sqsubseteq \text{hasPart} \rightsquigarrow \boxed{1}$$
$$\text{hasPart} \cdot \text{hasProperPart} \sqsubseteq \underline{\text{hasProperPart}}$$
$$\text{hasProperPart} \cdot \text{hasPart} \sqsubseteq \underline{\text{hasProperPart}}$$

$\prec$-Regularity

1. $R \cdot R \sqsubseteq R$
2. $R^{-} \sqsubseteq R$
3. $S_1 \cdots S_n \sqsubseteq R$
4. $R \cdot S_1 \cdots S_n \sqsubseteq R$
5. $S_1 \cdots S_n \cdot R \sqsubseteq R$

$\prec$-Regularity vs. Regularity

Example (hasProperPart $\prec$ hasPart)

$$\text{hasProperPart} \sqsubseteq \text{hasPart} \rightsquigarrow \boxed{3}$$
$$\text{hasPart} \cdot \text{hasPart} \sqsubseteq \text{hasPart} \rightsquigarrow \boxed{1}$$
$$\text{hasPart} \cdot \text{hasProperPart} \sqsubseteq \text{hasProperPart}$$
$$\text{hasProperPart} \cdot \text{hasPart} \sqsubseteq \text{hasProperPart}$$

$\prec$-Regularity

1. $R \cdot R \sqsubseteq R$
2. $R^{-} \sqsubseteq R$
3. $S_1 \cdots S_n \sqsubseteq R$
4. $R \cdot S_1 \cdots S_n \sqsubseteq R$
5. $S_1 \cdots S_n \cdot R \sqsubseteq R$

- The set of RIAs is not $\prec$-regular because of the cycle:

$$\text{hasProperPart} \sqsubseteq \text{hasPart}$$
$$\text{hasPart} \cdot [\ldots] \sqsubseteq \text{hasProperPart}$$

$\prec$-Regularity vs. Regularity

Example (hasProperPart $\prec$ hasPart)

$$\text{hasProperPart} \sqsubseteq \text{hasPart} \rightsquigarrow \boxed{3}$$
$$\text{hasPart} \cdot \text{hasPart} \sqsubseteq \text{hasPart} \rightsquigarrow \boxed{1}$$
$$\text{hasPart} \cdot \text{hasProperPart} \sqsubseteq \text{hasProperPart}$$
$$\text{hasProperPart} \cdot \text{hasPart} \sqsubseteq \text{hasProperPart}$$

$\prec$-Regularity

1. $R \cdot R \sqsubseteq R$
2. $R^{-} \sqsubseteq R$
3. $S_1 \cdots S_n \sqsubseteq R$
4. $R \cdot S_1 \cdots S_n \sqsubseteq R$
5. $S_1 \cdots S_n \cdot R \sqsubseteq R$

- The set of RIAs is not $\prec$-regular because of the cycle:

$$\text{hasProperPart} \sqsubseteq \text{hasPart}$$
$$\text{hasPart} \cdot [\ldots] \sqsubseteq \text{hasProperPart}$$

- However the languages induced by the RIAs are regular:

$$L(\text{hasPart}) = \{(\text{hasPart} \mid \text{hasProperPart})^{+}\}$$

$\prec$-Regularity vs. Regularity

Example (hasProperPart $\prec$ hasPart)

$$\text{hasProperPart} \sqsubseteq \text{hasPart} \rightsquigarrow 3$$
$$\text{hasPart} \cdot \text{hasPart} \sqsubseteq \text{hasPart} \rightsquigarrow 1$$
$$\text{hasPart} \cdot \text{hasProperPart} \sqsubseteq \text{hasProperPart}$$
$$\text{hasProperPart} \cdot \text{hasPart} \sqsubseteq \text{hasProperPart}$$

$\prec$-Regularity

1. $R \cdot R \sqsubseteq R$
2. $R^{-} \sqsubseteq R$
3. $S_1 \cdots S_n \sqsubseteq R$
4. $R \cdot S_1 \cdots S_n \sqsubseteq R$
5. $S_1 \cdots S_n \cdot R \sqsubseteq R$

- The set of RIAs is not $\prec$-regular because of the cycle:

$$\text{hasProperPart} \sqsubseteq \text{hasPart}$$
$$\text{hasPart} \cdot [\ldots] \sqsubseteq \text{hasProperPart}$$

- However the languages induced by the RIAs are regular:

$$L(\text{hasPart}) = \{(\text{hasPart} \mid \text{hasProperPart})^{+}\}$$
$$L(\text{hasProperPart}) = L(\text{hasPart}) \setminus \{\text{hasPart}^{*}\}$$

Other Use Cases

1. Complex RIAs in GALEN:

$$\mathsf{NonPartitivelyContaines} \sqsubseteq \mathsf{Contains}$$
$$\mathsf{Contains} \cdot \mathsf{Contains} \sqsubseteq \mathsf{Contains}$$
$$\mathsf{NonPartitivelyContains} \cdot \mathsf{Contains} \sqsubseteq \mathsf{NonPartitivelyContains}$$

Other Use Cases

1. Complex RIAs in GALEN:

$$\text{NonPartitivelyContaines} \sqsubseteq \text{Contains}$$
$$\text{Contains} \cdot \text{Contains} \sqsubseteq \text{Contains}$$
$$\text{NonPartitivelyContains} \cdot \text{Contains} \sqsubseteq \text{NonPartitivelyContains}$$

2. SEP Triplets encoding (used, e.g., in SNOMED CT):
   - Hand $\rightsquigarrow$ Hand_S, Hand_E, Hand_P
   - encoding using complex RIAs [Santisrivaraporn et.al 2007]:

   $\text{Hand}_S \equiv \exists\text{isPartOf}.\text{Hand}_E$
   $\text{Hand}_P \equiv \exists\text{isProperPartOf}.\text{Hand}_E$
   $\text{isPartOf} \cdot \text{isProperPartOf} \sqsubseteq \text{isProperPartOf} \ldots$

Other Use Cases

1. Complex RIAs in GALEN:

$$\mathsf{NonPartitivelyContaines} \sqsubseteq \mathsf{Contains}$$
$$\mathsf{Contains} \cdot \mathsf{Contains} \sqsubseteq \mathsf{Contains}$$
$$\mathsf{NonPartitivelyContains} \cdot \mathsf{Contains} \sqsubseteq \mathsf{NonPartitivelyContains}$$

2. SEP Triplets encoding (used, e.g., in SNOMED CT):
   - Hand $\rightsquigarrow$ Hand_S, Hand_E, Hand_P
   - encoding using complex RIAs [Santisrivaraporn et.al 2007]:

   $\mathsf{Hand}_S \equiv \exists\mathsf{isPartOf}.\mathsf{Hand}_E$
   $\mathsf{Hand}_P \equiv \exists\mathsf{isProperPartOf}.\mathsf{Hand}_E$
   $\mathsf{isPartOf} \cdot \mathsf{isProperPartOf} \sqsubseteq \mathsf{isProperPartOf} \ldots$

3. Roles describing “sibling” relations:

$$\mathsf{hasChild}^- \cdot \mathsf{hasChild} \sqsubseteq \mathsf{hasSibling}$$
$$\mathsf{hasSibling} \cdot \mathsf{hasSibling} \sqsubseteq \mathsf{hasSibling}$$
$$\mathsf{hasChild} \cdot \mathsf{hasSibling} \sqsubseteq \mathsf{hasChild}$$

CHARACTERIZING ALL REGULAR RIAS?

- In general, it is not possible to check whether a given set of RIAs induces a regular language:

THEOREM (WELL-KNOWN RESULT)

It is undecidable to check if a context free grammar defines a regular language.

CHARACTERIZING ALL REGULAR RIAS?

- In general, it is not possible to check whether a given set of RIAs induces a regular language:

THEOREM (WELL-KNOWN RESULT)

It is undecidable to check if a context free grammar defines a regular language.

- Even if the user supplies all regular automata for $L(R)$, it is not possible to check if they correspond to the given RIAs:

THEOREM (WELL-KNOWN RESULT)

It is undecidable to check if a context free grammar over Σ induces Σ^*.

Summary of the Main Results

We relax the restrictions on RIAs in $\mathcal{SROIQ}$ such that:

1. They are backward compatible with the original restrictions

Summary of the Main Results

We relax the restrictions on RIAs in $\mathcal{SROIQ}$ such that:

1. They are backward compatible with the original restrictions
2. Can be checked in polynomial time

Summary of the Main Results

We relax the restrictions on RIAs in $\mathcal{SROIQ}$ such that:

1. They are backward compatible with the original restrictions
2. Can be checked in polynomial time
3. Corresponding NFAs can be constructed in exponential time

Summary of the Main Results

We relax the restrictions on RIAs in $\mathcal{SROIQ}$ such that:

1. They are backward compatible with the original restrictions
2. Can be checked in polynomial time
3. Corresponding NFAs can be constructed in exponential time
4. For every regular set of RIAs there exists a conservative extension that satisfies our restrictions.

Outline

The Main Idea

Example

$S_1 \cdot R \sqsubseteq R$

$R \cdot R \sqsubseteq R$

$R \cdot S_2 \sqsubseteq R$

$L(R) = \{{S_1}^* \cdot R^+ \cdot {S_2}^*\}$

- Possible proofs for $S_1 \cdot R \cdot R \cdot S_2 \sqsubseteq^* R$:
  1. $S_1 \cdot R \ \cdot R \cdot S_2 \sqsubseteq$

The Main Idea

Example

- $S_1 \cdot R \sqsubseteq R$ $\quad L(R) = \{{S_1}^* \cdot R^+ \cdot {S_2}^*\}$
  $R \cdot R \sqsubseteq R$
  $R \cdot S_2 \sqsubseteq R$

- Possible proofs for $S_1 \cdot R \cdot R \cdot S_2 \sqsubseteq^* R$:
  1. $(S_1 \cdot R) \cdot R \cdot S_2 \sqsubseteq \underline{R} \cdot R \cdot S_2$

THE MAIN IDEA

EXAMPLE

$S_1 \cdot R \sqsubseteq R$ $\quad L(R) = \{S_1{}^* \cdot R^+ \cdot S_2{}^*\}$

➤ $R \cdot R \sqsubseteq R$

$R \cdot S_2 \sqsubseteq R$

- Possible proofs for $S_1 \cdot R \cdot R \cdot S_2 \sqsubseteq^* R$:
  1. $(S_1 \cdot R) \cdot R \cdot S_2 \sqsubseteq (R \cdot R) \cdot S_2 \sqsubseteq \underline{R} \cdot S_2$

The Main Idea

Example

$S_1 \cdot R \sqsubseteq R$ $\quad$ $L(R) = \{{S_1}^* \cdot R^+ \cdot {S_2}^*\}$

$R \cdot R \sqsubseteq R$

➤ $R \cdot S_2 \sqsubseteq R$

- Possible proofs for $S_1 \cdot R \cdot R \cdot S_2 \sqsubseteq^* R$:
  1. $(S_1 \cdot R) \cdot R \cdot S_2 \sqsubseteq (R \cdot R) \cdot S_2 \sqsubseteq R \cdot S_2 \sqsubseteq R$

The Main Idea

Example

$S_1 \cdot R \sqsubseteq R$

$R \cdot R \sqsubseteq R$

$R \cdot S_2 \sqsubseteq R$

$L(R) = \{{S_1}^* \cdot R^+ \cdot {S_2}^*\}$

- Possible proofs for $S_1 \cdot R \cdot R \cdot S_2 \sqsubseteq^* R$:
  1. $(S_1 \cdot R) \cdot R \cdot S_2 \sqsubseteq (R \cdot R) \cdot S_2 \sqsubseteq R \cdot S_2 \sqsubseteq R$
  2. $S_1 \cdot R \cdot (R \cdot S_2) \sqsubseteq S_1 \cdot (R \cdot R) \sqsubseteq S_1 \cdot R \sqsubseteq R$

The Main Idea

Example

$S_1 \cdot R \sqsubseteq R$ $\quad L(R) = \{{S_1}^* \cdot R^+ \cdot {S_2}^*\}$

$R \cdot R \sqsubseteq R$

$R \cdot S_2 \sqsubseteq R$

- Possible proofs for $S_1 \cdot R \cdot R \cdot S_2 \sqsubseteq^* R$:
  1. $(S_1 \cdot R) \cdot R \cdot S_2 \sqsubseteq (R \cdot R) \cdot S_2 \sqsubseteq R \cdot S_2 \sqsubseteq R$
  2. $S_1 \cdot R \cdot (R \cdot S_2) \sqsubseteq S_1 \cdot (R \cdot R) \sqsubseteq S_1 \cdot R \sqsubseteq R$
  3. $S_1 \cdot (R \cdot R) \cdot S_2 \sqsubseteq (S_1 \cdot R) \cdot S_2 \sqsubseteq R \cdot S_2 \sqsubseteq R$

The Main Idea

Example

$S_1 \cdot R \sqsubseteq R$

$R \cdot R \sqsubseteq R$

$R \cdot S_2 \sqsubseteq R$

$L(R) = \{S_1{}^* \cdot R^+ \cdot S_2{}^*\}$

- Possible proofs for $S_1 \cdot R \cdot R \cdot S_2 \sqsubseteq^* R$:
  1. $(S_1 \cdot R) \cdot R \cdot S_2 \sqsubseteq (R \cdot R) \cdot S_2 \sqsubseteq R \cdot S_2 \sqsubseteq R$
  2. $S_1 \cdot R \cdot (R \cdot S_2) \sqsubseteq S_1 \cdot (R \cdot R) \sqsubseteq S_1 \cdot R \sqsubseteq R$
  3. $S_1 \cdot (R \cdot R) \cdot S_2 \sqsubseteq (S_1 \cdot R) \cdot S_2 \sqsubseteq R \cdot S_2 \sqsubseteq R$
  4. $S_1 \cdot (R \cdot R) \cdot S_2 \sqsubseteq S_1 \cdot (R \cdot S_2) \sqsubseteq S_1 \cdot R \sqsubseteq R$

The Main Idea

Example

$S_1 \cdot R \sqsubseteq R$

$R \cdot R \sqsubseteq R$

$R \cdot S_2 \sqsubseteq R$

$L(R) = \{{S_1}^* \cdot R^+ \cdot {S_2}^*\}$

- Possible proofs for $S_1 \cdot R \cdot R \cdot S_2 \sqsubseteq^* R$:
  1. $(S_1 \cdot R) \cdot R \cdot S_2 \sqsubseteq (R \cdot R) \cdot S_2 \sqsubseteq R \cdot S_2 \sqsubseteq R$
  2. $S_1 \cdot R \cdot (R \cdot S_2) \sqsubseteq S_1 \cdot (R \cdot R) \sqsubseteq S_1 \cdot R \sqsubseteq R$
  3. $S_1 \cdot (R \cdot R) \cdot S_2 \sqsubseteq (S_1 \cdot R) \cdot S_2 \sqsubseteq R \cdot S_2 \sqsubseteq R$
  4. $S_1 \cdot (R \cdot R) \cdot S_2 \sqsubseteq S_1 \cdot (R \cdot S_2) \sqsubseteq S_1 \cdot R \sqsubseteq R$

$\Rightarrow$ Does not depend on the order of the rule applications

The Main Idea

Example

$S_1 \cdot R \sqsubseteq R$

$R \cdot R \sqsubseteq R$

$R \cdot S_2 \sqsubseteq R$

$L(R) = \{{S_1}^* \cdot R^+ \cdot {S_2}^*\}$

- Possible proofs for $S_1 \cdot R \cdot R \cdot S_2 \sqsubseteq^* R$:
  1. $(S_1 \cdot R) \cdot R \cdot S_2 \sqsubseteq (R \cdot R) \cdot S_2 \sqsubseteq R \cdot S_2 \sqsubseteq R$
  2. $S_1 \cdot R \cdot (R \cdot S_2) \sqsubseteq S_1 \cdot (R \cdot R) \sqsubseteq S_1 \cdot R \sqsubseteq R$
  3. $S_1 \cdot (R \cdot R) \cdot S_2 \sqsubseteq (S_1 \cdot R) \cdot S_2 \sqsubseteq R \cdot S_2 \sqsubseteq R$
  4. $S_1 \cdot (R \cdot R) \cdot S_2 \sqsubseteq S_1 \cdot (R \cdot S_2) \sqsubseteq S_1 \cdot R \sqsubseteq R$

  $\Rightarrow$ Does not depend on the order of the rule applications
- In particular, for every role chains ρ_1, ρ_2 over S_1, S_2, and R:

$\rho_1 \cdot R \cdot \rho_2 \sqsubseteq^* R$ implies $(\rho_1 \cdot R) \cdot \rho_2 \sqsubseteq^* R \cdot \rho_2 \sqsubseteq^* R$

and $\rho_1 \cdot (R \cdot \rho_2) \sqsubseteq^* \rho_1 \cdot R \sqsubseteq^* R$

Generalizing Regularity Conditions

- Instead of strict order $\prec$ we use a preorder $\precsim$, thus allowing equivalent roles $R_1 \approx R_2$

Generalizing Regularity Conditions

- Instead of strict order $\prec$ we use a preorder $\precsim$, thus allowing equivalent roles $R_1 \approx R_2$
- Admissibility conditions:
  1. $R \approx R^-$
  2. $\rho_1 \cdot S \cdot \rho_2 \sqsubseteq R$ implies $S \precsim R$
  3. if $\rho_1 \cdot R_1 \cdot \rho_2 \sqsubseteq^* R_2$ and $R_1 \approx R_2$ then:
     $(\rho_1 \cdot R_1) \cdot \rho_2 \sqsubseteq^* R_3 \cdot \rho_2 \sqsubseteq^* R$, and
     $\rho_1 \cdot (R_1 \cdot \rho_2) \sqsubseteq^* \rho_1 \cdot R_4 \sqsubseteq^* R$.
     In this case we say that $\rho_1 \cdot R_1 \cdot \rho_2 \sqsubseteq^* R$ is stratified.

Generalizing Regularity Conditions

- Instead of strict order $\prec$ we use a preorder $\precsim$, thus allowing equivalent roles $R_1 \approx R_2$
- Admissibility conditions:
  1. $R \approx R^-$
  2. $\rho_1 \cdot S \cdot \rho_2 \sqsubseteq R$ implies $S \precsim R$
  3. if $\rho_1 \cdot R_1 \cdot \rho_2 \sqsubseteq^* R_2$ and $R_1 \approx R_2$ then:
     $(\rho_1 \cdot R_1) \cdot \rho_2 \sqsubseteq^* R_3 \cdot \rho_2 \sqsubseteq^* R$, and
     $\rho_1 \cdot (R_1 \cdot \rho_2) \sqsubseteq^* \rho_1 \cdot R_4 \sqsubseteq^* R$.
     In this case we say that $\rho_1 \cdot R_1 \cdot \rho_2 \sqsubseteq^* R$ is stratified.
- If a set of RIAs satisfy 1 – 3 then it is called stratified.

Checking Stratified RIAs

Lemma

It is possible to check in polynomial time if $\rho_1 \cdot R_1 \cdot \rho_2 \sqsubseteq^* R$ *is stratified.*

Checking Stratified RIAs

Lemma

It is possible to check in polynomial time if $\rho_1 \cdot R_1 \cdot \rho_2 \sqsubseteq^* R$ *is stratified.*

Proof.

1. Find R_3 such that $\rho_1 \cdot R_1 \sqsubseteq^* R_3$ and $R_3 \cdot \rho_2 \sqsubseteq^* R$
2. Find R_4 such that $R_1 \cdot \rho_2 \sqsubseteq^* R_4$ and $\rho_1 \cdot R_4 \sqsubseteq^* R$

Checking if $\rho \sqsubseteq^* R$, equivalently, $\rho \in L(R)$ is polynomial (membership problem for context-free languages) □

Checking Stratified Sets of RIAs

Requires to check that all $\rho_1 \cdot R_1 \cdot \rho_2 \sqsubseteq^* R$ is stratified for infinitely many implied RIAs.

Checking Stratified Sets of RIAs

Requires to check that all $\rho_1 \cdot R_1 \cdot \rho_2 \sqsubseteq^* R$ is stratified for infinitely many implied RIAs.

Definition

RIAs $\rho_1 \cdot R_1 \sqsubseteq R_2$ and $R_3 \cdot \rho_2 \sqsubseteq R_4$ overlap if $R_2 \sqsubseteq^* R_3$.

Checking Stratified Sets of RIAs

Requires to check that all $\rho_1 \cdot R_1 \cdot \rho_2 \sqsubseteq^* R$ is stratified for infinitely many implied RIAs.

Definition

RIAs $\rho_1 \cdot R_1 \sqsubseteq R_2$ and $R_3 \cdot \rho_2 \sqsubseteq R_4$ overlap if $R_2 \sqsubseteq^* R_3$.
In this case we say that $\rho_1 \cdot R_1 \cdot \rho_2 \sqsubseteq^* R_4$ is the overlap.

Checking Stratified Sets of RIAs

Requires to check that all $\rho_1 \cdot R_1 \cdot \rho_2 \sqsubseteq^* R$ is stratified for infinitely many implied RIAs.

Definition

RIAs $\rho_1 \cdot R_1 \sqsubseteq R_2$ and $R_3 \cdot \rho_2 \sqsubseteq R_4$ overlap if $R_2 \sqsubseteq^* R_3$.
In this case we say that $\rho_1 \cdot R_1 \cdot \rho_2 \sqsubseteq^* R_4$ is the overlap.

Lemma

Let $\mathcal{R}$ be a set of RIAs and $\bar{\mathcal{R}}$ be obtained from $\mathcal{R}$ by adding $\rho^- \sqsubseteq R^-$ for every $\rho \sqsubseteq R \in \mathcal{R}$. Then $\mathcal{R}$ is stratified iff:

1. *every $\rho_1 \cdot R_1 \cdot \rho_2 \sqsubseteq R \in \bar{\mathcal{R}}$ is stratified, and*
2. *every overlap $\rho_1 \cdot R_1 \cdot \rho_2 \sqsubseteq R$ of RIAs in $\bar{\mathcal{R}}$ is stratified.*

Checking Stratified Sets of RIAs

Requires to check that all $\rho_1 \cdot R_1 \cdot \rho_2 \sqsubseteq^* R$ is stratified for infinitely many implied RIAs.

Definition

RIAs $\rho_1 \cdot R_1 \sqsubseteq R_2$ and $R_3 \cdot \rho_2 \sqsubseteq R_4$ overlap if $R_2 \sqsubseteq^* R_3$.
In this case we say that $\rho_1 \cdot R_1 \cdot \rho_2 \sqsubseteq^* R_4$ is the overlap.

Lemma

Let $\mathcal{R}$ be a set of RIAs and $\bar{\mathcal{R}}$ be obtained from $\mathcal{R}$ by adding $\rho^- \sqsubseteq R^-$ for every $\rho \sqsubseteq R \in \mathcal{R}$. Then $\mathcal{R}$ is stratified iff:

1. *every $\rho_1 \cdot R_1 \cdot \rho_2 \sqsubseteq R \in \bar{\mathcal{R}}$ is stratified, and*
2. *every overlap $\rho_1 \cdot R_1 \cdot \rho_2 \sqsubseteq R$ of RIAs in $\bar{\mathcal{R}}$ is stratified.*

Corollary

It is possible to check in polynomial time whether $\mathcal{R}$ is stratified.

Example 1

1. $\text{hasProperPart} \sqsubseteq \text{hasPart}$
2. $\text{hasPart} \cdot \text{hasPart} \sqsubseteq \text{hasPart}$
3. $\text{hasPart} \cdot \text{hasProperPart} \sqsubseteq \text{hasProperPart}$
4. $\text{hasProperPart} \cdot \text{hasPart} \sqsubseteq \text{hasProperPart}$

EXAMPLE 1

1. hasProperPart $\sqsubseteq$ hasPart
2. hasPart $\cdot$ hasPart $\sqsubseteq$ hasPart
3. hasPart $\cdot$ hasProperPart $\sqsubseteq$ hasProperPart
4. hasProperPart $\cdot$ hasPart $\sqsubseteq$ hasProperPart

- overlap between 2 and 2:

hasPart $\cdot$ hasPart $\cdot$ hasPart $\sqsubseteq$ hasPart

Example 1

1. hasProperPart $\sqsubseteq$ hasPart
2. hasPart $\cdot$ hasPart $\sqsubseteq$ hasPart
3. hasPart $\cdot$ hasProperPart $\sqsubseteq$ hasProperPart
4. hasProperPart $\cdot$ hasPart $\sqsubseteq$ hasProperPart

- overlap between 2 and 2:

$$(\text{hasPart} \cdot \text{hasPart}) \cdot \text{hasPart} \sqsubseteq \text{hasPart}$$
$$\text{hasPart} \cdot \text{hasPart} \sqsubseteq \text{hasPart}$$

Example 1

1. hasProperPart $\sqsubseteq$ hasPart
2. hasPart $\cdot$ hasPart $\sqsubseteq$ hasPart
3. hasPart $\cdot$ hasProperPart $\sqsubseteq$ hasProperPart
4. hasProperPart $\cdot$ hasPart $\sqsubseteq$ hasProperPart

- overlap between 2 and 2:

hasPart $\cdot$ (hasPart $\cdot$ hasPart) $\sqsubseteq$ hasPart

hasPart $\cdot$ hasPart $\sqsubseteq$ hasPart

EXAMPLE 1

1. hasProperPart $\sqsubseteq$ hasPart
2. hasPart · hasPart $\sqsubseteq$ hasPart
3. hasPart · hasProperPart $\sqsubseteq$ hasProperPart
4. hasProperPart · hasPart $\sqsubseteq$ hasProperPart

- overlap between 2 and 2 is stratified;

Example 1

1. hasProperPart $\sqsubseteq$ hasPart
2. hasPart · hasPart $\sqsubseteq$ hasPart
3. hasPart · hasProperPart $\sqsubseteq$ hasProperPart
4. hasProperPart · hasPart $\sqsubseteq$ hasProperPart

- overlap between 2 and 2 is stratified;
- overlap between 2 and 3:

hasPart · hasPart · hasProperPart $\sqsubseteq$ hasProperPart

Example 1

1. hasProperPart $\sqsubseteq$ hasPart
2. hasPart $\cdot$ hasPart $\sqsubseteq$ hasPart
3. hasPart $\cdot$ hasProperPart $\sqsubseteq$ hasProperPart
4. hasProperPart $\cdot$ hasPart $\sqsubseteq$ hasProperPart

- overlap between 2 and 2 is stratified;
- overlap between 2 and 3:

(hasPart $\cdot$ hasPart) $\cdot$ hasProperPart $\sqsubseteq$ hasProperPart

hasPart $\cdot$ hasProperPart $\sqsubseteq$ hasProperPart

Example 1

1. hasProperPart $\sqsubseteq$ hasPart
2. hasPart $\cdot$ hasPart $\sqsubseteq$ hasPart
3. hasPart $\cdot$ hasProperPart $\sqsubseteq$ hasProperPart
4. hasProperPart $\cdot$ hasPart $\sqsubseteq$ hasProperPart

- overlap between 2 and 2 is stratified;
- overlap between 2 and 3:

hasPart $\cdot$ (hasPart $\cdot$ hasProperPart) $\sqsubseteq$ hasProperPart

hasPart $\cdot$ hasProperPart $\sqsubseteq$ hasProperPart

Example 1

1. hasProperPart $\sqsubseteq$ hasPart
2. hasPart $\cdot$ hasPart $\sqsubseteq$ hasPart
3. hasPart $\cdot$ hasProperPart $\sqsubseteq$ hasProperPart
4. hasProperPart $\cdot$ hasPart $\sqsubseteq$ hasProperPart

- overlap between 2 and 2 is stratified;
- overlap between 2 and 3 is stratified;

Example 1

1. hasProperPart $\sqsubseteq$ hasPart
2. hasPart · hasPart $\sqsubseteq$ hasPart
3. hasPart · hasProperPart $\sqsubseteq$ hasProperPart
4. hasProperPart · hasPart $\sqsubseteq$ hasProperPart

- overlap between 2 and 2 is stratified;
- overlap between 2 and 3 is stratified;
- overlap between 2 and 4:

hasProperPart · hasPart · hasPart $\sqsubseteq$ hasProperPart

EXAMPLE 1

1. hasProperPart $\sqsubseteq$ hasPart
2. hasPart · hasPart $\sqsubseteq$ hasPart
3. hasPart · hasProperPart $\sqsubseteq$ hasProperPart
4. hasProperPart · hasPart $\sqsubseteq$ hasProperPart

- overlap between 2 and 2 is stratified;
- overlap between 2 and 3 is stratified;
- overlap between 2 and 4:

(hasProperPart · hasPart) · hasPart $\sqsubseteq$ hasProperPart
hasProperPart · hasPart $\sqsubseteq$ hasProperPart

Example 1

1. hasProperPart $\sqsubseteq$ hasPart
2. hasPart $\cdot$ hasPart $\sqsubseteq$ hasPart
3. hasPart $\cdot$ hasProperPart $\sqsubseteq$ hasProperPart
4. hasProperPart $\cdot$ hasPart $\sqsubseteq$ hasProperPart

- overlap between 2 and 2 is stratified;
- overlap between 2 and 3 is stratified;
- overlap between 2 and 4:

hasProperPart $\cdot$ (hasPart $\cdot$ hasPart) $\sqsubseteq$ hasProperPart
hasProperPart $\cdot$ hasPart $\sqsubseteq$ hasProperPart

EXAMPLE 1

1. hasProperPart $\sqsubseteq$ hasPart
2. hasPart $\cdot$ hasPart $\sqsubseteq$ hasPart
3. hasPart $\cdot$ hasProperPart $\sqsubseteq$ hasProperPart
4. hasProperPart $\cdot$ hasPart $\sqsubseteq$ hasProperPart

- overlap between 2 and 2 is stratified;
- overlap between 2 and 3 is stratified;
- overlap between 2 and 4 is stratified;

EXAMPLE 1

1. hasProperPart $\sqsubseteq$ hasPart
2. hasPart · hasPart $\sqsubseteq$ hasPart
3. hasPart · hasProperPart $\sqsubseteq$ hasProperPart
4. hasProperPart · hasPart $\sqsubseteq$ hasProperPart

- overlap between 2 and 2 is stratified;
- overlap between 2 and 3 is stratified;
- overlap between 2 and 4 is stratified;
- overlap between 3 and 3:

hasPart · hasPart · hasProperPart $\sqsubseteq$ hasProperPart

Example 1

1. hasProperPart $\sqsubseteq$ hasPart
2. hasPart · hasPart $\sqsubseteq$ hasPart
3. hasPart · hasProperPart $\sqsubseteq$ hasProperPart
4. hasProperPart · hasPart $\sqsubseteq$ hasProperPart

- overlap between 2 and 2 is stratified;
- overlap between 2 and 3 is stratified;
- overlap between 2 and 4 is stratified;
- overlap between 3 and 3:

(hasPart · hasPart) · hasProperPart $\sqsubseteq$ hasProperPart
hasPart · hasProperPart $\sqsubseteq$ hasProperPart

Example 1

1. hasProperPart $\sqsubseteq$ hasPart
2. hasPart · hasPart $\sqsubseteq$ hasPart
3. hasPart · hasProperPart $\sqsubseteq$ hasProperPart
4. hasProperPart · hasPart $\sqsubseteq$ hasProperPart

- overlap between 2 and 2 is stratified;
- overlap between 2 and 3 is stratified;
- overlap between 2 and 4 is stratified;
- overlap between 3 and 3:

hasPart · (hasPart · hasProperPart) $\sqsubseteq$ hasProperPart
hasPart · hasProperPart $\sqsubseteq$ hasProperPart

Example 1

1. hasProperPart $\sqsubseteq$ hasPart
2. hasPart · hasPart $\sqsubseteq$ hasPart
3. hasPart · hasProperPart $\sqsubseteq$ hasProperPart
4. hasProperPart · hasPart $\sqsubseteq$ hasProperPart

- overlap between 2 and 2 is stratified;
- overlap between 2 and 3 is stratified;
- overlap between 2 and 4 is stratified;
- overlap between 3 and 3 is stratified;

Example 1

1. hasProperPart $\sqsubseteq$ hasPart
2. hasPart $\cdot$ hasPart $\sqsubseteq$ hasPart
3. hasPart $\cdot$ hasProperPart $\sqsubseteq$ hasProperPart
4. hasProperPart $\cdot$ hasPart $\sqsubseteq$ hasProperPart

- overlap between 2 and 2 is stratified;
- overlap between 2 and 3 is stratified;
- overlap between 2 and 4 is stratified;
- overlap between 3 and 3 is stratified;
- overlap between 3 and 4:

hasPart $\cdot$ hasProperPart $\cdot$ hasPart $\sqsubseteq$ hasProperPart

Example 1

1. hasProperPart $\sqsubseteq$ hasPart
2. hasPart · hasPart $\sqsubseteq$ hasPart
3. hasPart · hasProperPart $\sqsubseteq$ hasProperPart
4. hasProperPart · hasPart $\sqsubseteq$ hasProperPart

- overlap between 2 and 2 is stratified;
- overlap between 2 and 3 is stratified;
- overlap between 2 and 4 is stratified;
- overlap between 3 and 3 is stratified;
- overlap between 3 and 4:

(hasPart · hasProperPart) · hasPart $\sqsubseteq$ hasProperPart

hasProperPart · hasPart $\sqsubseteq$ hasProperPart

Example 1

1. hasProperPart $\sqsubseteq$ hasPart
2. hasPart · hasPart $\sqsubseteq$ hasPart
3. hasPart · hasProperPart $\sqsubseteq$ hasProperPart
4. hasProperPart · hasPart $\sqsubseteq$ hasProperPart

- overlap between 2 and 2 is stratified;
- overlap between 2 and 3 is stratified;
- overlap between 2 and 4 is stratified;
- overlap between 3 and 3 is stratified;
- overlap between 3 and 4:

hasPart · (hasProperPart · hasPart) $\sqsubseteq$ hasProperPart

hasPart · hasProperPart $\sqsubseteq$ hasProperPart

EXAMPLE 1

1. $\text{hasProperPart} \sqsubseteq \text{hasPart}$
2. $\text{hasPart} \cdot \text{hasPart} \sqsubseteq \text{hasPart}$
3. $\text{hasPart} \cdot \text{hasProperPart} \sqsubseteq \underline{\text{hasProperPart}}$
4. $\underline{\text{hasProperPart}} \cdot \text{hasPart} \sqsubseteq \text{hasProperPart}$

- overlap between 2 and 2 is stratified;
- overlap between 2 and 3 is stratified;
- overlap between 2 and 4 is stratified;
- overlap between 3 and 3 is stratified;
- overlap between 3 and 4 is stratified;

Example 1

1. hasProperPart $\sqsubseteq$ hasPart
2. hasPart · hasPart $\sqsubseteq$ hasPart
3. hasPart · <u>hasProperPart</u> $\sqsubseteq$ hasProperPart
4. hasProperPart · hasPart $\sqsubseteq$ <u>hasProperPart</u>

- overlap between 2 and 2 is stratified;
- overlap between 2 and 3 is stratified;
- overlap between 2 and 4 is stratified;
- overlap between 3 and 3 is stratified;
- overlap between 3 and 4 is stratified;
- another overlap between 3 and 4:

hasPart · hasProperPart · hasPart $\sqsubseteq$ hasProperPart

Example 1

1. hasProperPart $\sqsubseteq$ hasPart
2. hasPart $\cdot$ hasPart $\sqsubseteq$ hasPart
3. hasPart $\cdot$ hasProperPart $\sqsubseteq$ hasProperPart
4. hasProperPart $\cdot$ hasPart $\sqsubseteq$ hasProperPart

- overlap between 2 and 2 is stratified;
- overlap between 2 and 3 is stratified;
- overlap between 2 and 4 is stratified;
- overlap between 3 and 3 is stratified;
- overlap between 3 and 4 is stratified;
- another overlap between 3 and 4:

(hasPart $\cdot$ hasProperPart) $\cdot$ hasPart $\sqsubseteq$ hasProperPart

hasProperPart $\cdot$ hasPart $\sqsubseteq$ hasProperPart

Example 1

1. hasProperPart $\sqsubseteq$ hasPart
2. hasPart · hasPart $\sqsubseteq$ hasPart
3. hasPart · hasProperPart $\sqsubseteq$ hasProperPart
4. hasProperPart · hasPart $\sqsubseteq$ hasProperPart

- overlap between 2 and 2 is stratified;
- overlap between 2 and 3 is stratified;
- overlap between 2 and 4 is stratified;
- overlap between 3 and 3 is stratified;
- overlap between 3 and 4 is stratified;
- another overlap between 3 and 4:

hasPart · (hasProperPart · hasPart) $\sqsubseteq$ hasProperPart
hasPart · hasProperPart $\sqsubseteq$ hasProperPart

Example 1

1. hasProperPart $\sqsubseteq$ hasPart
2. hasPart $\cdot$ hasPart $\sqsubseteq$ hasPart
3. hasPart $\cdot$ hasProperPart $\sqsubseteq$ hasProperPart
4. hasProperPart $\cdot$ hasPart $\sqsubseteq$ hasProperPart

- overlap between 2 and 2 is stratified;
- overlap between 2 and 3 is stratified;
- overlap between 2 and 4 is stratified;
- overlap between 3 and 3 is stratified;
- overlap between 3 and 4 is stratified;
- another overlap between 3 and 4 is stratified;

Example 1

1. hasProperPart $\sqsubseteq$ hasPart
2. hasPart · hasPart $\sqsubseteq$ hasPart
3. hasPart · hasProperPart $\sqsubseteq$ hasProperPart
4. hasProperPart · hasPart $\sqsubseteq$ hasProperPart

- overlap between 2 and 2 is stratified;
- overlap between 2 and 3 is stratified;
- overlap between 2 and 4 is stratified;
- overlap between 3 and 3 is stratified;
- overlap between 3 and 4 is stratified;
- another overlap between 3 and 4 is stratified;
- overlap between 4 and 4:

hasProperPart · hasPart · hasPart $\sqsubseteq$ hasProperPart

Example 1

1. hasProperPart $\sqsubseteq$ hasPart
2. hasPart · hasPart $\sqsubseteq$ hasPart
3. hasPart · hasProperPart $\sqsubseteq$ hasProperPart
4. hasProperPart · hasPart $\sqsubseteq$ hasProperPart

- overlap between 2 and 2 is stratified;
- overlap between 2 and 3 is stratified;
- overlap between 2 and 4 is stratified;
- overlap between 3 and 3 is stratified;
- overlap between 3 and 4 is stratified;
- another overlap between 3 and 4 is stratified;
- overlap between 4 and 4:

(hasProperPart · hasPart) · hasPart $\sqsubseteq$ hasProperPart

hasProperPart · hasPart $\sqsubseteq$ hasProperPart

Example 1

1. hasProperPart $\sqsubseteq$ hasPart
2. hasPart · hasPart $\sqsubseteq$ hasPart
3. hasPart · hasProperPart $\sqsubseteq$ hasProperPart
4. hasProperPart · hasPart $\sqsubseteq$ hasProperPart

- overlap between 2 and 2 is stratified;
- overlap between 2 and 3 is stratified;
- overlap between 2 and 4 is stratified;
- overlap between 3 and 3 is stratified;
- overlap between 3 and 4 is stratified;
- another overlap between 3 and 4 is stratified;
- overlap between 4 and 4:

hasProperPart · (hasPart · hasPart) $\sqsubseteq$ hasProperPart
hasProperPart · hasPart $\sqsubseteq$ hasProperPart

EXAMPLE 1

1. hasProperPart $\sqsubseteq$ hasPart
2. hasPart · hasPart $\sqsubseteq$ hasPart
3. hasPart · hasProperPart $\sqsubseteq$ hasProperPart
4. hasProperPart · hasPart $\sqsubseteq$ hasProperPart

- overlap between 2 and 2 is stratified;
- overlap between 2 and 3 is stratified;
- overlap between 2 and 4 is stratified;
- overlap between 3 and 3 is stratified;
- overlap between 3 and 4 is stratified;
- another overlap between 3 and 4 is stratified;
- overlap between 4 and 4 is stratified;

Example 1

1. hasProperPart $\sqsubseteq$ hasPart
2. hasPart · hasPart $\sqsubseteq$ hasPart
3. hasPart · hasProperPart $\sqsubseteq$ hasProperPart
4. hasProperPart · hasPart $\sqsubseteq$ hasProperPart

- overlap between 2 and 2 is stratified;
- overlap between 2 and 3 is stratified;
- overlap between 2 and 4 is stratified;
- overlap between 3 and 3 is stratified;
- overlap between 3 and 4 is stratified;
- another overlap between 3 and 4 is stratified;
- overlap between 4 and 4 is stratified;
- all overlaps are stratified

Example 2

1. $\mathsf{hasChild}^- \cdot \mathsf{hasChild} \sqsubseteq \mathsf{hasSibling}$
2. $\mathsf{hasSibling} \cdot \mathsf{hasSibling} \sqsubseteq \mathsf{hasSibling}$
3. $\mathsf{hasChild} \cdot \mathsf{hasSibling} \sqsubseteq \mathsf{hasChild}$

Example 2

1. $\mathsf{hasChild}^- \cdot \mathsf{hasChild} \sqsubseteq \mathsf{hasSibling}$
2. $\mathsf{hasSibling} \cdot \mathsf{hasSibling} \sqsubseteq \mathsf{hasSibling}$
3. $\mathsf{hasChild} \cdot \mathsf{hasSibling} \sqsubseteq \mathsf{hasChild}$

- overlap between 1 and 2:
  $\mathsf{hasChild}^- \cdot \mathsf{hasChild} \cdot \mathsf{hasSibling} \sqsubseteq \mathsf{hasSibling}$

Example 2

1. $\text{hasChild}^- \cdot \text{hasChild} \sqsubseteq \underline{\text{hasSibling}}$
2. $\underline{\text{hasSibling}} \cdot \text{hasSibling} \sqsubseteq \text{hasSibling}$
3. $\text{hasChild} \cdot \text{hasSibling} \sqsubseteq \text{hasChild}$

- overlap between 1 and 2:

$$(\text{hasChild}^- \cdot \text{hasChild}) \cdot \text{hasSibling} \sqsubseteq \text{hasSibling}$$

$$\text{hasSibling} \cdot \text{hasSibling} \sqsubseteq \text{hasSibling}$$

Example 2

1. $\text{hasChild}^{-} \cdot \text{hasChild} \sqsubseteq \underline{\text{hasSibling}}$
2. $\underline{\text{hasSibling}} \cdot \text{hasSibling} \sqsubseteq \text{hasSibling}$
3. $\text{hasChild} \cdot \text{hasSibling} \sqsubseteq \text{hasChild}$

- overlap between 1 and 2:

$$\text{hasChild}^{-} \cdot (\text{hasChild} \cdot \text{hasSibling}) \sqsubseteq \text{hasSibling}$$
$$\text{hasChild}^{-} \cdot \text{hasChild} \sqsubseteq \text{hasSibling}$$

Example 2

1. $\mathsf{hasChild}^- \cdot \mathsf{hasChild} \sqsubseteq \mathsf{hasSibling}$
2. $\mathsf{hasSibling} \cdot \mathsf{hasSibling} \sqsubseteq \mathsf{hasSibling}$
3. $\mathsf{hasChild} \cdot \mathsf{hasSibling} \sqsubseteq \mathsf{hasChild}$

- overlap between 1 and 2 is stratified;

Example 2

1 $\text{hasChild}^- \cdot \text{hasChild} \sqsubseteq \text{hasSibling}$

2 $\text{hasSibling} \cdot \text{hasSibling} \sqsubseteq \text{hasSibling}$

3 $\text{hasChild} \cdot \text{hasSibling} \sqsubseteq \text{hasChild}$

- overlap between 1 and 2 is stratified;
- another overlap between 1 and 2:
  $\text{hasSibling} \cdot \text{hasChild}^- \cdot \text{hasChild} \sqsubseteq \text{hasSibling}$

Example 2

1. $\text{hasChild}^{-} \cdot \text{hasChild} \sqsubseteq \underline{\text{hasSibling}}$
2. $\text{hasSibling} \cdot \underline{\text{hasSibling}} \sqsubseteq \text{hasSibling}$
3. $\text{hasChild} \cdot \text{hasSibling} \sqsubseteq \text{hasChild}$

- overlap between 1 and 2 is stratified;
- another overlap between 1 and 2:

$$(\text{hasSibling} \cdot \text{hasChild}^{-}) \cdot \text{hasChild} \sqsubseteq \text{hasSibling}$$
$$? \cdot \text{hasChild} \sqsubseteq \text{hasSibling}$$

EXAMPLE 2

1. $\mathsf{hasChild}^- \cdot \mathsf{hasChild} \sqsubseteq \mathsf{hasSibling}$
2. $\mathsf{hasSibling} \cdot \underline{\mathsf{hasSibling}} \sqsubseteq \mathsf{hasSibling}$
3. $\mathsf{hasChild} \cdot \mathsf{hasSibling} \sqsubseteq \mathsf{hasChild}$

- overlap between 1 and 2 is stratified;
- another overlap between 1 and 2:

$$(\mathsf{hasSibling} \cdot \mathsf{hasChild}^-) \cdot \mathsf{hasChild} \sqsubseteq \mathsf{hasSibling}$$
$$\mathsf{hasChild}^- \cdot \mathsf{hasChild} \sqsubseteq \mathsf{hasSibling}$$

Example 2

1. $\text{hasChild}^- \cdot \text{hasChild} \sqsubseteq \text{hasSibling}$
2. $\text{hasSibling} \cdot \text{hasSibling} \sqsubseteq \text{hasSibling}$
3. $\text{hasChild} \cdot \text{hasSibling} \sqsubseteq \text{hasChild}$
4. $\text{hasSibling} \sqsubseteq \text{hasSibling}^-$

- overlap between 1 and 2 is stratified;
- another overlap between 1 and 2:

$$(\text{hasSibling} \cdot \text{hasChild}^-) \cdot \text{hasChild} \sqsubseteq \text{hasSibling}$$
$$\text{hasChild}^- \cdot \text{hasChild} \sqsubseteq \text{hasSibling}$$

Example 2

1 $\mathsf{hasChild}^- \cdot \mathsf{hasChild} \sqsubseteq \underline{\mathsf{hasSibling}}$

2 $\mathsf{hasSibling} \cdot \underline{\mathsf{hasSibling}} \sqsubseteq \mathsf{hasSibling}$

3 $\mathsf{hasChild} \cdot \mathsf{hasSibling} \sqsubseteq \mathsf{hasChild}$

4 $\mathsf{hasSibling} \sqsubseteq \mathsf{hasSibling}^-$

- overlap between 1 and 2 is stratified;
- another overlap between 1 and 2:

$$\mathsf{hasSibling} \cdot (\mathsf{hasChild}^- \cdot \mathsf{hasChild}) \sqsubseteq \mathsf{hasSibling}$$
$$\mathsf{hasChild} \cdot \mathsf{hasSibling} \sqsubseteq \mathsf{hasSibling}$$

Example 2

1. $\mathsf{hasChild}^- \cdot \mathsf{hasChild} \sqsubseteq \mathsf{hasSibling}$
2. $\mathsf{hasSibling} \cdot \mathsf{hasSibling} \sqsubseteq \mathsf{hasSibling}$
3. $\mathsf{hasChild} \cdot \mathsf{hasSibling} \sqsubseteq \mathsf{hasChild}$
4. $\mathsf{hasSibling} \sqsubseteq \mathsf{hasSibling}^-$

- overlap between 1 and 2 is stratified;
- another overlap between 1 and 2 is stratified;

Example 2

1. $\text{hasChild}^- \cdot \text{hasChild} \sqsubseteq \underline{\text{hasSibling}}$
2. $\text{hasSibling} \cdot \text{hasSibling} \sqsubseteq \text{hasSibling}$
3. $\text{hasChild} \cdot \underline{\text{hasSibling}} \sqsubseteq \text{hasChild}$
4. $\text{hasSibling} \sqsubseteq \text{hasSibling}^-$

- overlap between 1 and 2 is stratified;
- another overlap between 1 and 2 is stratified;
- overlap between 1 and 3:
  $\text{hasChild} \cdot \text{hasChild}^- \cdot \text{hasChild} \sqsubseteq \text{hasChild}$

EXAMPLE 2

1. $\mathsf{hasChild}^{-} \cdot \mathsf{hasChild} \sqsubseteq \underline{\mathsf{hasSibling}}$
2. $\mathsf{hasSibling} \cdot \mathsf{hasSibling} \sqsubseteq \mathsf{hasSibling}$
3. $\mathsf{hasChild} \cdot \underline{\mathsf{hasSibling}} \sqsubseteq \mathsf{hasChild}$
4. $\mathsf{hasSibling} \sqsubseteq \mathsf{hasSibling}^{-}$

- overlap between 1 and 2 is stratified;
- another overlap between 1 and 2 is stratified;
- overlap between 1 and 3:

$$(\mathsf{hasChild} \cdot \mathsf{hasChild}^{-}) \cdot \mathsf{hasChild} \sqsubseteq \mathsf{hasChild}$$
$$? \qquad \cdot \mathsf{hasChild} \sqsubseteq \mathsf{hasChild}$$

Example 2

1 $\text{hasChild}^- \cdot \text{hasChild} \sqsubseteq \underline{\text{hasSibling}}$

2 $\text{hasSibling} \cdot \text{hasSibling} \sqsubseteq \text{hasSibling}$

3 $\text{hasChild} \cdot \underline{\text{hasSibling}} \sqsubseteq \text{hasChild}$

4 $\text{hasSibling} \sqsubseteq \text{hasSibling}^-$

- overlap between 1 and 2 is stratified;
- another overlap between 1 and 2 is stratified;
- overlap between 1 and 3:

$$(\text{hasChild} \cdot \text{hasChild}^-) \cdot \text{hasChild} \sqsubseteq \text{hasChild}$$
$$\text{hasPartner} \cdot \text{hasChild} \sqsubseteq \text{hasChild}$$

EXAMPLE 2

1. $\text{hasChild}^- \cdot \text{hasChild} \sqsubseteq \text{hasSibling}$
2. $\text{hasSibling} \cdot \text{hasSibling} \sqsubseteq \text{hasSibling}$
3. $\text{hasChild} \cdot \text{hasSibling} \sqsubseteq \text{hasChild}$
4. $\text{hasSibling} \sqsubseteq \text{hasSibling}^-$
5. $\text{hasChild} \cdot \text{hasChild}^- \sqsubseteq \text{hasPartner}$
6. $\text{hasPartner} \cdot \text{hasPartner} \sqsubseteq \text{hasPartner}$
7. $\text{hasPartner} \cdot \text{hasChild} \sqsubseteq \text{hasChild}$
8. $\text{hasPartner} \sqsubseteq \text{hasPartner}^-$

- overlap between 1 and 2 is stratified;
- another overlap between 1 and 2 is stratified;
- overlap between 1 and 3:

$$(\text{hasChild} \cdot \text{hasChild}^-) \cdot \text{hasChild} \sqsubseteq \text{hasChild}$$
$$\text{hasPartner} \cdot \text{hasChild} \sqsubseteq \text{hasChild}$$

EXAMPLE 2

1. $\text{hasChild}^- \cdot \text{hasChild} \sqsubseteq \text{hasSibling}$
2. $\text{hasSibling} \cdot \text{hasSibling} \sqsubseteq \text{hasSibling}$
3. $\text{hasChild} \cdot \text{hasSibling} \sqsubseteq \text{hasChild}$
4. $\text{hasSibling} \sqsubseteq \text{hasSibling}^-$
5. $\text{hasChild} \cdot \text{hasChild}^- \sqsubseteq \text{hasPartner}$
6. $\text{hasPartner} \cdot \text{hasPartner} \sqsubseteq \text{hasPartner}$
7. $\text{hasPartner} \cdot \text{hasChild} \sqsubseteq \text{hasChild}$
8. $\text{hasPartner} \sqsubseteq \text{hasPartner}^-$

- overlap between 1 and 2 is stratified;
- another overlap between 1 and 2 is stratified;
- overlap between 1 and 3 is stratified;

Example 2

1. $\text{hasChild}^{-} \cdot \text{hasChild} \sqsubseteq \text{hasSibling}$
2. $\text{hasSibling} \cdot \text{hasSibling} \sqsubseteq \text{hasSibling}$
3. $\text{hasChild} \cdot \text{hasSibling} \sqsubseteq \text{hasChild}$
4. $\text{hasSibling} \sqsubseteq \text{hasSibling}^{-}$
5. $\text{hasChild} \cdot \text{hasChild}^{-} \sqsubseteq \text{hasPartner}$
6. $\text{hasPartner} \cdot \text{hasPartner} \sqsubseteq \text{hasPartner}$
7. $\text{hasPartner} \cdot \text{hasChild} \sqsubseteq \text{hasChild}$
8. $\text{hasPartner} \sqsubseteq \text{hasPartner}^{-}$

- overlap between 1 and 2 is stratified;
- another overlap between 1 and 2 is stratified;
- overlap between 1 and 3 is stratified;
- all remaining overlaps are stratified

EXAMPLE 2

1 $\mathsf{hasChild}^- \cdot \mathsf{hasChild} \sqsubseteq \mathsf{hasSibling}$

2 $\mathsf{hasSibling} \cdot \mathsf{hasSibling} \sqsubseteq \mathsf{hasSibling}$

3 $\mathsf{hasChild} \cdot \mathsf{hasSibling} \sqsubseteq \mathsf{hasChild}$

4 $\mathsf{hasSibling} \sqsubseteq \mathsf{hasSibling}^-$

5 $\mathsf{hasChild} \cdot \mathsf{hasChild}^- \sqsubseteq \mathsf{hasPartner}$

6 $\mathsf{hasPartner} \cdot \mathsf{hasPartner} \sqsubseteq \mathsf{hasPartner}$

7 $\mathsf{hasPartner} \cdot \mathsf{hasChild} \sqsubseteq \mathsf{hasChild}$

8 $\mathsf{hasPartner} \sqsubseteq \mathsf{hasPartner}^-$

- overlap between 1 and 2 is stratified;
- another overlap between 1 and 2 is stratified;
- overlap between 1 and 3 is stratified;
- all remaining overlaps are stratified

THEOREM

Every regular set of RIAs can be conservatively extended to stratified one by adding new RIAs.

Conclusions

New restrictions on complex RIAs:

- Backward compatible with the original restrictions
- Can be checked in polynomial time
- Imply regularity for RIAs ▸ Details
- NFAs can be constructed in exponential time
  $\Rightarrow$ computationaly optimal complexity for $\mathcal{SROIQ}$
- Can capture any regular compositional properties
- Can be used to discover missing RIAs

Stratified RIAs and Regularity

If $\rho \sqsubseteq^* R$ then it has a stratified proof:

Stratified RIAs and Regularity

If $\rho \sqsubseteq^* R$ then it has a stratified proof:

$\rho \sqsubseteq^* \quad \rho_0 \cdot R_1 \quad \cdot \rho_1 \cdot R_2 \cdot \rho_2 \cdots R_n \cdot \rho_n$ (using $\rho' \sqsubseteq S$ with $S \prec R$)

Stratified RIAs and Regularity

If $\rho \sqsubseteq^* R$ then it has a stratified proof:

$$\rho \sqsubseteq^* (\rho_0 \cdot R_1) \cdot \rho_1 \cdot R_2 \cdot \rho_2 \cdots R_n \cdot \rho_n \qquad \text{(using } \rho' \sqsubseteq S \text{ with } S \prec R\text{)}$$

$$\sqsubseteq^* \underline{R_1'} \cdot \rho_1 \cdot R_2 \cdot \rho_2 \cdots R_n \cdot \rho_n$$

Stratified RIAs and Regularity

If $\rho \sqsubseteq^* R$ then it has a stratified proof:

$$\begin{aligned} \rho &\sqsubseteq^* (\rho_0 \cdot R_1) \cdot \rho_1 \cdot R_2 \cdot \rho_2 \cdots R_n \cdot \rho_n \\ &\sqsubseteq^* (R_1' \cdot \rho_1 \cdot R_2) \cdot \rho_2 \cdots R_n \cdot \rho_n \\ &\sqsubseteq^* \underline{R_2'} \cdot \rho_2 \cdots R_n \cdot \rho_n \end{aligned}$$

(using $\rho' \sqsubseteq S$ with $S \prec R$)

Stratified RIAs and Regularity

If $\rho \sqsubseteq^* R$ then it has a stratified proof:

$$\begin{aligned}
\rho \sqsubseteq^* & (\rho_0 \cdot R_1) \cdot \rho_1 \cdot R_2 \cdot \rho_2 \cdots R_n \cdot \rho_n \\
\sqsubseteq^* & (R'_1 \cdot \rho_1 \cdot R_2) \cdot \rho_2 \cdots R_n \cdot \rho_n \\
\sqsubseteq^* & (R'_2 \cdot \rho_2 \cdots R_n) \cdot \rho_n \\
& \ldots \\
\sqsubseteq^* & \underline{R'_n} \cdot \rho_n
\end{aligned}$$

(using $\rho' \sqsubseteq S$ with $S \prec R$)

Stratified RIAs and Regularity

If $\rho \sqsubseteq^* R$ then it has a stratified proof:

$$
\begin{aligned}
\rho &\sqsubseteq^* (\rho_0 \cdot R_1) \cdot \rho_1 \cdot R_2 \cdot \rho_2 \cdots R_n \cdot \rho_n \qquad \text{(using } \rho' \sqsubseteq S \text{ with } S \prec R\text{)} \\
&\sqsubseteq^* (R_1' \cdot \rho_1 \cdot R_2) \cdot \rho_2 \cdots R_n \cdot \rho_n \\
&\sqsubseteq^* (R_2' \cdot \rho_2 \cdots R_n) \cdot \rho_n \\
&\ldots \\
&\sqsubseteq^* (R_n' \cdot \rho_n) \\
&\sqsubseteq^* \underline{R}
\end{aligned}
$$

Stratified RIAs and Regularity

If $\rho \sqsubseteq^* R$ then it has a stratified proof:

$$
\begin{aligned}
\rho \sqsubseteq^* & (\rho_0 \cdot R_1) \cdot \rho_1 \cdot R_2 \cdot \rho_2 \cdots R_n \cdot \rho_n \qquad \text{(using } \rho' \sqsubseteq S \text{ with } S \prec R\text{)} \\
\sqsubseteq^* & (R'_1 \cdot \rho_1 \cdot R_2) \cdot \rho_2 \cdots R_n \cdot \rho_n \\
\sqsubseteq^* & (R'_2 \cdot \rho_2 \cdots R_n) \cdot \rho_n \\
& \cdots \\
\sqsubseteq^* & (R'_n \cdot \rho_n) \\
\sqsubseteq^* & R
\end{aligned}
$$

Regularity follows from the fact that:

1. Left-linear context-free languages are regular
2. Regular languages are closed under substitutions