$\mathcal{SRIQ}$ and $\mathcal{SROIQ}$ are Harder than $\mathcal{SHOIQ}$

Yevgeny Kazakov

(presented by Birte Glimm)

Oxford University Computing Laboratory

May 15, 2008

Outline

Summary of the Main Results

Known Results (see DL Complexity Navigator[1])

(Finite model) reasoning is:

- ExpTime-complete for $\mathcal{SHIQ}$
- NExpTime-complete for $\mathcal{SHOIQ}$

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- 2ExpTime-hard for $\mathcal{SRIQ}$ [and even for $\mathcal{SR}$]
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In short: $\mathcal{H} \Rightarrow \mathcal{R}$ causes an exponential blowup!

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[2006] $\mathcal{SROIQ} = \mathcal{SRIQ} + \mathcal{SHOIQ}$

proposed as a basis for $\mathcal{OWL}$ **2** (a.k.a. $\mathcal{OWL}$ **1.1**)

Regular RIAs

- Integration of new constructions into existing tableau-based procedures:
- U, $\neg R(a,b)$, $Sym(R)$, $Ref(R)$, $Asy(S)$, $Irr(R)$, $Disj(S_1,S_2)$
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Example

$S \circ R \circ S \sqsubseteq R$ — not regular

$R_i \circ R_i \sqsubseteq R_{i+1}$ — regular by **3**
when $R_0 \prec R_1 \prec \cdots \prec R_n$

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Tableau: The Exponential Blowup

- Every regular RBox $\mathcal{R}$ induces a regular language:

$$L_{\mathcal{R}}(R) = \{S_1 S_2 \ldots S_n \mid S_1 \circ S_2 \circ \cdots \circ S_n \sqsubseteq^*_{\mathcal{R}} R\}$$

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$S \circ R \circ S \sqsubseteq R$ $\quad L_{\mathcal{R}}(R) = \{S^i R S^i \mid i \geq 0\}$ — non regular

$R_i \circ R_i \sqsubseteq R_{i+1}$ $\quad L_{\mathcal{R}}(R_{i+1}) = \{R_{i+1}\} \cup L_{\mathcal{R}}(R_i) \cdot L_{\mathcal{R}}(R_i)$

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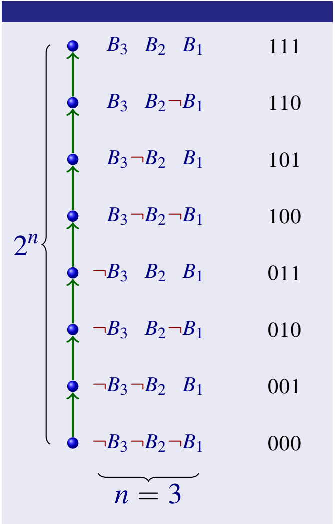
- Unfortunately $|L_{\mathcal{R}}(R)|$ can be exponential in $|\mathcal{R}|$: by induction on i one can show that $|L_{\mathcal{R}}(R_i)| \geq 2^i$
- This causes an exponential blowup compared to the procedure for $\mathcal{SHOIQ}$ $\quad \Leftarrow$ Unavoidable??

Outline

Exponential Chains in $\mathcal{ALC}$

- Integer counting technique:
  - A counter between 0 and $2^n - 1$ uses n concepts $B_1, \ldots, B_n$
  - The i-th bit of the counter corresponds to the value of B_i
  - The counter is incremented over R



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- Expressing in $\mathcal{ALC}$:

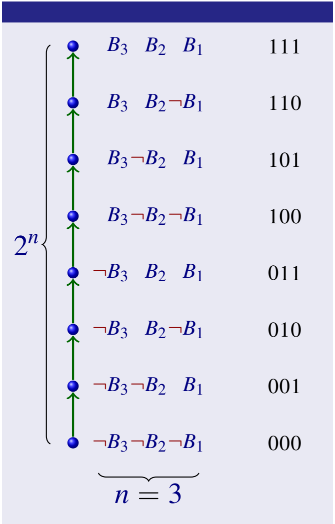
$Z \equiv \neg B_n \sqcap \cdots \sqcap \neg B_1$ — "Zero"

$E \equiv B_n \sqcap \cdots \sqcap B_1$ — "End"

$\neg E \equiv \exists R.\top$ — Successors

$\top \equiv (B_1 \sqcap \forall R.\neg B_1) \sqcup (\neg B_1 \sqcap \forall R.B_1)$
— The lowest bit always flips

$B_{i-1} \sqcap \forall R.\neg B_{i-1} \equiv$
$(B_i \sqcap \forall R.\neg B_i) \sqcup (\neg B_i \sqcap \forall R.B_i)$
— The bit flips if the lower bit changes from 1 to 0



DOUBLY-EXPONENTIAL CHAINS IN $\mathcal{SRIQ}$

- Encode the counter on exponentially-long chains
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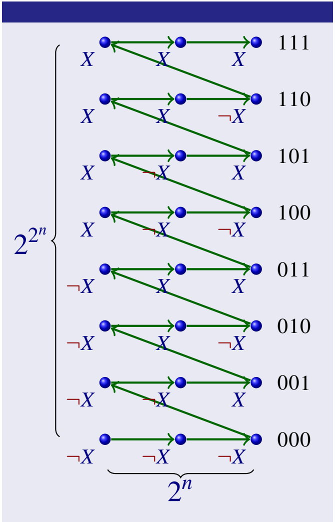
  - The chains are connected by "last-to-first element"



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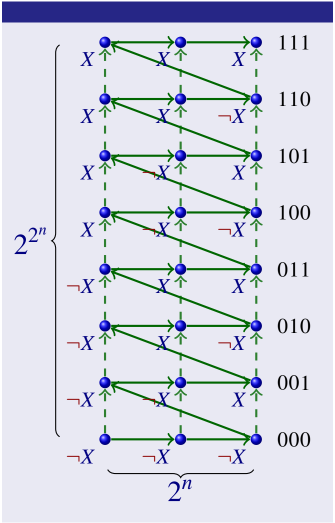

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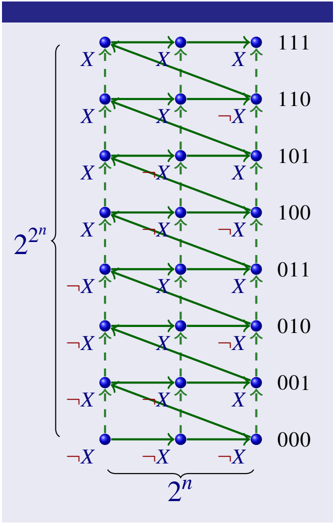

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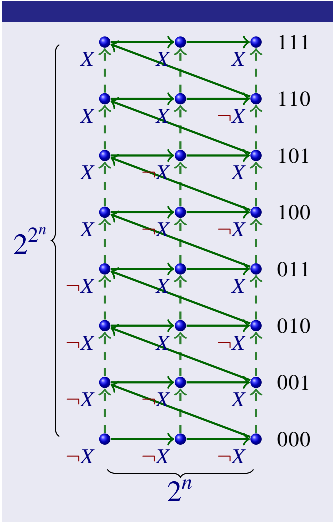

HARDNESS RESULT FOR $\mathcal{SROIQ}$

- The key idea is like in the NExpTime-hardness for $\mathcal{SHOIQ}$.
- In $\mathcal{SHOIQ}$ it is possible to express an exponential grid:

- Use two counters to encode the coordinates of the grid
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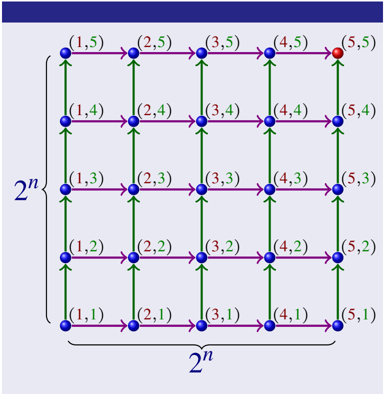

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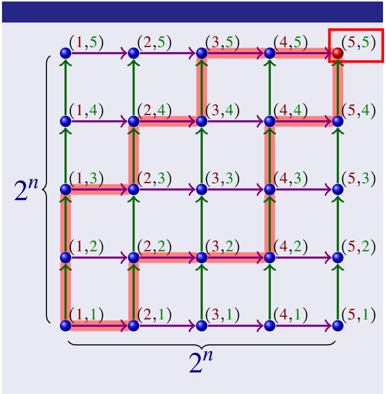

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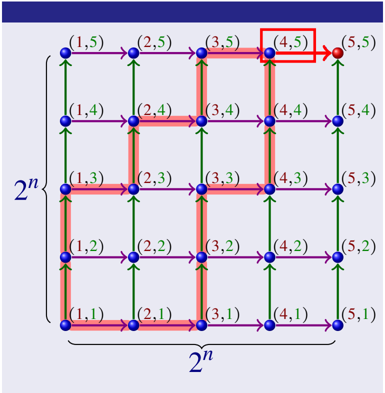

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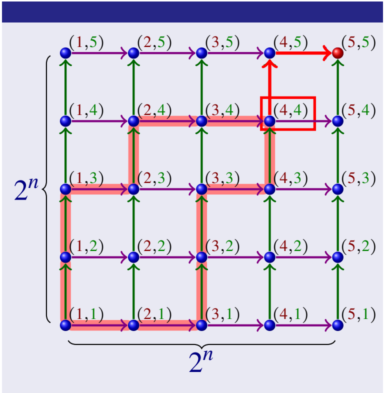

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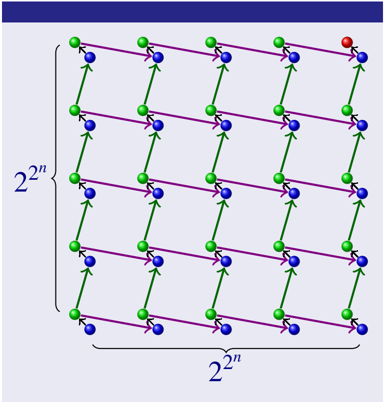

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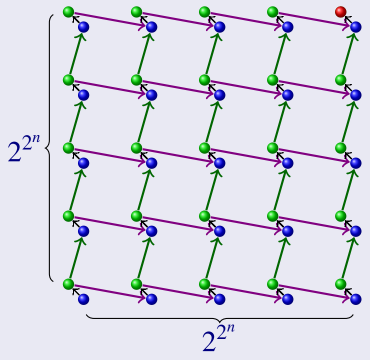
THEOREM

(Finite model) reasoning in $\mathcal{SROIQ}$ is N2ExpTime-hard. The result holds already for inverse functional roles and nominals.



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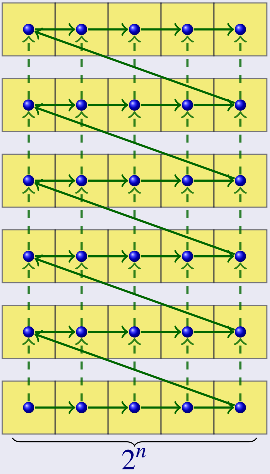
Hardness Result for $\mathcal{SRIQ}$

By reduction from the word problem for an exponential-space alternating Turing machine:

- Configurations are encoded on exponential chains
- Corresponding cells of successive configurations are connected by R_n
- Easy to simulate the computation

Computation of TM



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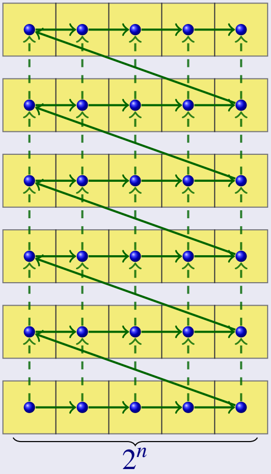
- Since $\text{AExpSpace} = \text{2ExpTime}$ we have:

Theorem

(Finite model) reasoning in $\mathcal{SRIQ}$ is 2ExpTime-hard. The result holds already without inverses and counting.

Computation of TM



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THE MEMBERSHIP RESULT FOR $\mathcal{SROIQ}$

The matching N2ExpTime upper bound for $\mathcal{SROIQ}$ is obtained by an exponential translation into $\mathcal{C}^2$:

Summary:

1. Simplify ontology to contain only axioms of forms 1–10
2. Eliminate axioms of form 10 using NFA
3. Translate the other axioms into $\mathcal{C}^2$

	Axiom	First-Order Translation
1	$A \sqsubseteq \forall r.B$	$\forall x.(A(x) \rightarrow \forall y.[r(x,y) \rightarrow B(y)])$
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4	$A \equiv \exists s.Self$	$\forall x.(A(x) \leftrightarrow s(x,x))$
5	$A_a \equiv \{a\}$	$\exists^{=1} y.A_a(y)$
6	$\bigsqcap A_i \sqsubseteq \bigsqcup B_j$	$\forall x.(\bigvee \neg A_i(x) \vee \bigvee B_j(x))$
7	$Disj(s_1, s_2)$	$\forall xy.(s_1(x,y) \wedge s_2(x,y) \rightarrow \bot)$
8	$s_1 \sqsubseteq s_2$	$\forall xy.(s_1(x,y) \rightarrow s_2(x,y))$
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10	$r_1 \circ \cdots \circ r_n \sqsubseteq v$,	$n \geq 1$, v is non-simple

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Key property for step 2

Axioms of form 10 can interact only with axioms of form 1, since other axioms contain only simple roles $s_{(i)}$

Elimination of Complex RIAs

The main idea

"Absorb" regular RIAs into axioms of the form $A \sqsubseteq \forall r.B$

- For each $A \sqsubseteq \forall r.B$, complex RIAs induce properties:
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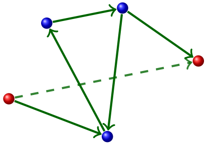

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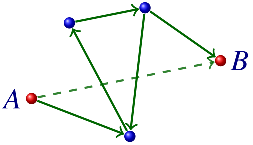

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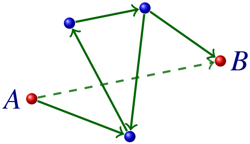

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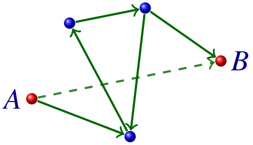

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- Note that $|Q|$ can be exponential in $|\mathcal{R}|$!

THE MEMBERSHIP RESULT FOR $\mathcal{SROIQ}$

The matching N2ExpTime upper bound for $\mathcal{SROIQ}$ is obtained by an exponential translation into $\mathcal{C}^2$:

Summary:

1. Simplify ontology to contain only axioms of forms 1–10 (polynom.)
2. Eliminate axioms of form 10 using NFA (exponential step!)
3. Translate the other axioms into $\mathcal{C}^2$ (NExpTime-complete)

	Axiom	First-Order Translation
1	$A \sqsubseteq \forall r.B$	$\forall x.(A(x) \rightarrow \forall y.[r(x,y) \rightarrow B(y)])$
2	$A \sqsubseteq \geqslant n\, s.B$	$\forall x.(A(x) \rightarrow \exists^{\geq n} y.[s(x,y) \wedge B(y)])$
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5	$A_a \equiv \{a\}$	$\exists^{=1} y.A_a(y)$
6	$\bigsqcap A_i \sqsubseteq \bigsqcup B_j$	$\forall x.(\bigvee \neg A_i(x) \vee \bigvee B_j(x))$
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THEOREM (UPPER COMPLEXITY FOR $\mathcal{SROIQ}$)

(Finite model) reasoning in $\mathcal{SROIQ}$ is N2ExpTime

Outline

1 Introduction

2 Hardness Results

3 Membership Results

4 Discussion

SUMMARY

- We have identified exact computational complexity of $\mathcal{SROIQ}$ to be N2ExpTime; $\mathcal{SRIQ}$ is 2ExpTime-hard.
- Complexity blowup is due to complex RIAs $R_1 \circ \cdots \circ R_n \sqsubseteq R$, in particular because they can chain a fixed exponential number of roles
- Explains the exponential blowup in the tableau procedures for $\mathcal{SRIQ}$ and $\mathcal{SROIQ}$

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- Explains the exponential blowup in the tableau procedures for $\mathcal{SRIQ}$ and $\mathcal{SROIQ}$
- Open problems:
  1. Upper bound for $\mathcal{SRIQ}$? Conjecture: 2ExpTime
  2. Upper & Lower bounds for $\mathcal{RIQ}$? Conjecture: 2ExpTime

  $\mathcal{RIQ}$ allows only for restricted complex RIAs of the form $R \circ S \sqsubseteq R$ and $S \circ R \sqsubseteq R$ which cannot be used in our constructions

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- Only the size of the RBox has a high complexity impact:

<table>
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Questions?

- Please send difficult questions to

Yevgeny Kazakov

yevgeny.kazakov@comlab.ox.ac.uk

- Our contribution:
  1. $\mathcal{SROIQ}$ [$\mathcal{SROIF}$] is N2ExpTime-complete
  2. $\mathcal{SRIQ}$ [$\mathcal{SR}$] is 2ExpTime-hard

- Thank you for your attention!