$\mathcal{RIQ}$ and $\mathcal{SROIQ}$ are Harder than $\mathcal{SHOIQ}$

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(presented by Birte Glimm)

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Outline

Summary of the Main Results

Known Results (See DL Complexity Navigator[1])

(Finite model) reasoning is:

- ExpTime-complete for $\mathcal{SHIQ}$
- NExpTime-complete for $\mathcal{SHOIQ}$

[1] http://www.cs.man.ac.uk/~ezolin/dl/

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(Finite model) reasoning is:

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Theorem (New Results in This Talk)

(Finite model) reasoning is:

- 2ExpTime-hard for $\mathcal{SRIQ}$ [$\mathcal{RIQ}$, and even for $\mathcal{R}$]
- N2ExpTime-complete for $\mathcal{SROIQ}$ [and for $\mathcal{SROIF}$]

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Timeline: From $\mathcal{SHIQ}$ to $\mathcal{SROIQ}$

[2003] $\mathcal{SHIQ}$ was extended to $\mathcal{RIQ}$ with complex RIAs:

- $R \circ S \sqsubseteq R$ (left-linear)
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1. $R \circ R \sqsubseteq R$ (transitivity)
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[2006] $\mathcal{SROIQ} = \mathcal{SRIQ} + \mathcal{SHOIQ}$

currently being standardized by W3C as the basis of $\mathcal{OWL}$ **2**—the Ontology Web Language v. 2

REGULAR RIAS

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Example

$S \circ R \circ S \sqsubseteq R$ — not regular

$S \circ S \sqsubseteq R$ — regular by 3 when $S \prec R$

Tableau: The Exponential Blowup

- Every regular RBox $\mathcal{R}$ induces a regular language:

$$L_{\mathcal{R}}(R) = \{S_1 S_2 \ldots S_n \mid S_1 \circ S_2 \circ \cdots \circ S_n \sqsubseteq^*_{\mathcal{R}} R\}$$

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EXAMPLE (CONTINUED)

1. $S \circ R \circ S \sqsubseteq R$ $\quad L_{\mathcal{R}}(R) = \{S^i\, R\, S^i \mid i \geq 0\}$ — non regular
2. $R_i \circ R_i \sqsubseteq R_{i+1}$ $\quad L_{\mathcal{R}}(R_{i+1}) = \{R_{i+1}\} \cup L_{\mathcal{R}}(R_i) \cdot L_{\mathcal{R}}(R_i)$ — regular (because finite)

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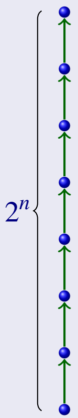
- This causes an exponential blowup in the tableau procedure
- Can one avoid this blowup?

– Our results imply that is not possible!

Outline

EXPONENTIAL CHAINS IN $\mathcal{ALC}$

- Well-known "integer counter" technique:



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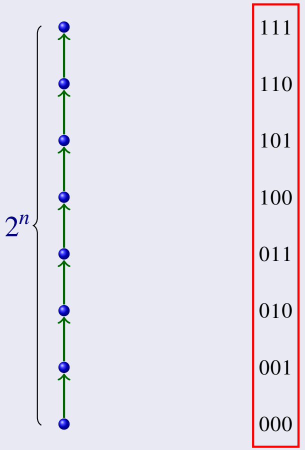

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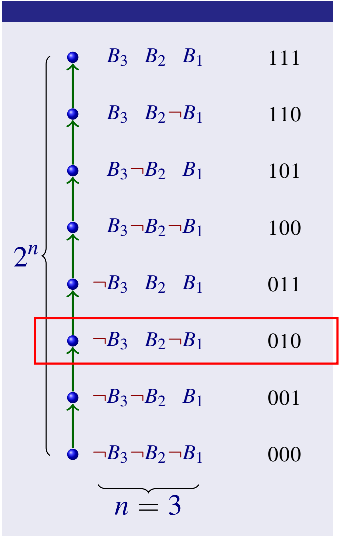

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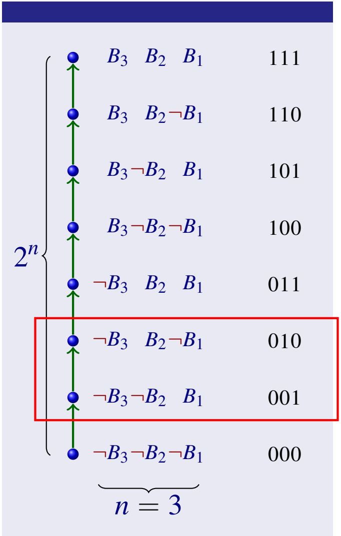
- The counter is incremented over R:

  The bit is flipped iff all the preceding bits = 1



DOUBLY-EXPONENTIAL CHAINS IN $\mathcal{SRIQ}$

- Encode the counter on exponentially-long chains
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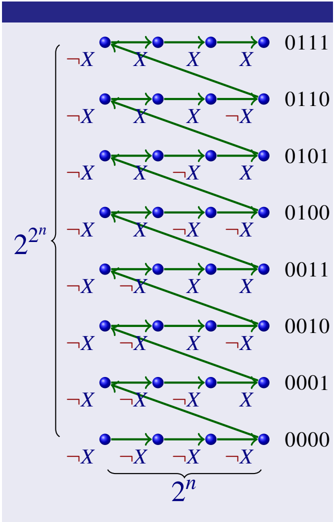

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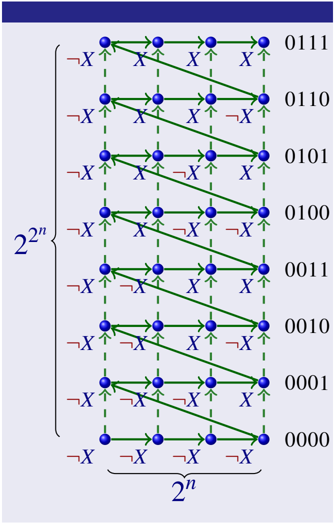

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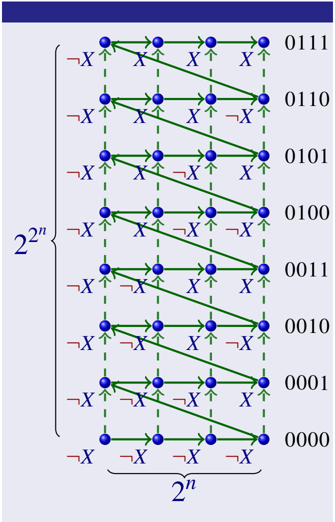
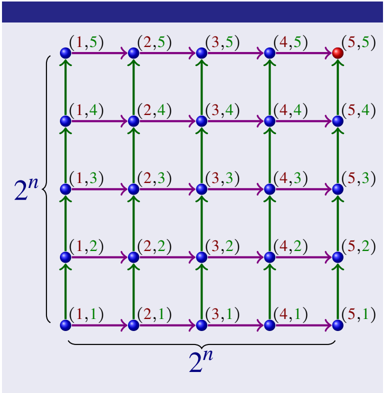
  - $R_i \circ R_i \sqsubseteq R_{i+1}$, $0 \leq i \leq n$
  - Complex RIAs connect elements reachable over exactly 2^n roles:
  - $\underbrace{R \circ R \circ \cdots \circ R}_{k} \sqsubseteq R_n$ iff $k = 2^n$



The Hardness Result for $\mathcal{SROIQ}$

- The key idea is like in the NExpTime-hardness for $\mathcal{SHOIQ}$.
- In $\mathcal{SHOIQ}$ it is possible to express an exponential grid:

- Use two counters to encode the coordinates of the grid
- Increment / copy the counters over respective roles



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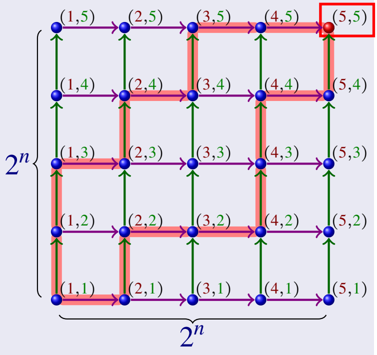

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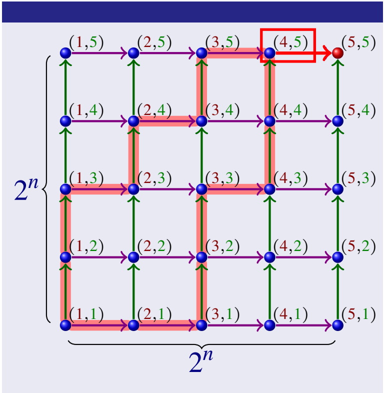

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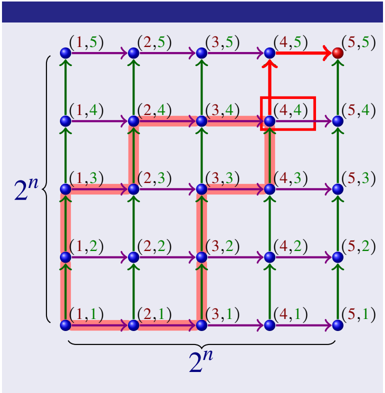

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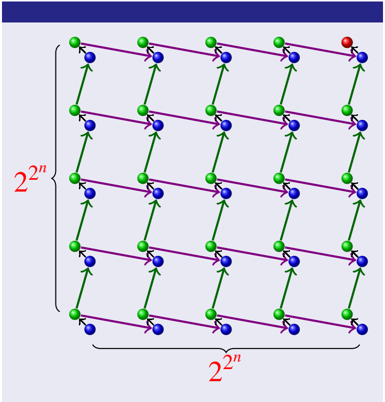

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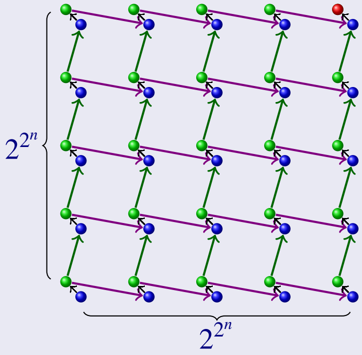
Theorem

(Finite model) reasoning in $\mathcal{SROIQ}$ is N2ExpTime-hard. The result holds already for inverse functional roles and nominals.



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By reduction from the word problem for an exponential-space alternating Turing machine:

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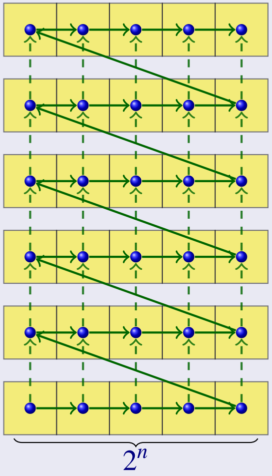
- Corresponding cells of successive configurations are connected by R_n
- Easy to simulate the computation

Computation of TM



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- Since $\text{AExpSpace} = \text{2ExpTime}$ we have:

Theorem

(Finite model) reasoning in $\mathcal{SRIQ}$ is 2ExpTime-hard. The result holds already without inverse roles and counting.

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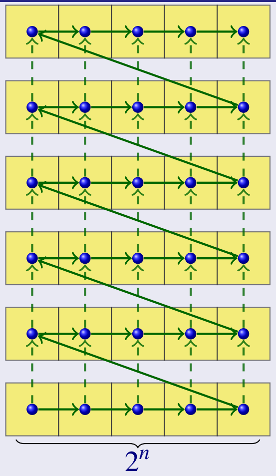

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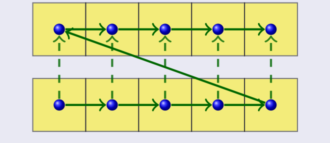

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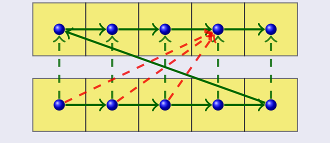

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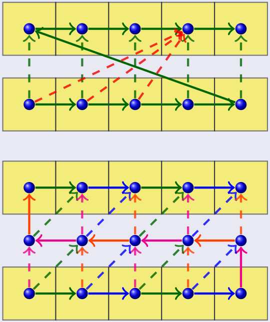

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THE MEMBERSHIP RESULT FOR $\mathcal{SROIQ}$

The matching N2ExpTime upper bound for $\mathcal{SROIQ}$ is obtained by an exponential translation into $\mathcal{C}^2$:

Summary:

1. Simplify ontology to contain only axioms of forms 1–10
2. Eliminate axioms of form 10 using NFA
3. Translate the other axioms into $\mathcal{C}^2$

	Axiom	First-Order Translation
1	$A \sqsubseteq \forall r.B$	$\forall x.(A(x) \rightarrow \forall y.[r(x,y) \rightarrow B(y)])$
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5	$A_a \equiv \{a\}$	$\exists^{=1} y.A_a(y)$
6	$\bigsqcap A_i \sqsubseteq \bigsqcup B_j$	$\forall x.(\bigvee \neg A_i(x) \vee \bigvee B_j(x))$
7	$Disj(s_1, s_2)$	$\forall xy.(s_1(x,y) \wedge s_2(x,y) \rightarrow \bot)$
8	$s_1 \sqsubseteq s_2$	$\forall xy.(s_1(x,y) \rightarrow s_2(x,y))$
9	$s_1 \sqsubseteq s_2^-$	$\forall xy.(s_1(x,y) \rightarrow s_2(y,x))$
10	$r_1 \circ \cdots \circ r_n \sqsubseteq v,\quad n \geq 1,\ v$ is non-simple	

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9	$s_1 \sqsubseteq s_2^-$	$\forall xy.(s_1(x,y) \rightarrow s_2(y,x))$
10	$r_1 \circ \cdots \circ r_n \sqsubseteq \underline{v}$,	$n \geq 1$, $\underline{v}$ is non-simple

Key property for step 2

Axioms of form 10 can interact only with axioms of form 1, since other axioms contain only simple roles $s_{(i)}$

The Membership Result for $\mathcal{SROIQ}$

The matching N2ExpTime upper bound for $\mathcal{SROIQ}$ is obtained by an exponential translation into $\mathcal{C}^2$:

Summary:

1. Simplify ontology to contain only axioms of forms 1–10 (polynom.)
2. Eliminate axioms of form 10 using NFA (exponential step!)
3. Translate the other axioms into $\mathcal{C}^2$ (NExpTime-complete)

	Axiom	First-Order Translation
1	$A \sqsubseteq \forall r.B$	$\forall x.(A(x) \rightarrow \forall y.[r(x,y) \rightarrow B(y)])$
2	$A \sqsubseteq \geqslant n\, s.B$	$\forall x.(A(x) \rightarrow \exists^{\geq n} y.[s(x,y) \wedge B(y)])$
3	$A \sqsubseteq \leqslant n\, s.B$	$\forall x.(A(x) \rightarrow \exists^{\leq n} y.[s(x,y) \wedge B(y)])$
4	$A \equiv \exists s.Self$	$\forall x.(A(x) \leftrightarrow s(x,x))$
5	$A_a \equiv \{a\}$	$\exists^{=1} y.A_a(y)$
6	$\bigsqcap A_i \sqsubseteq \bigsqcup B_j$	$\forall x.(\bigvee \neg A_i(x) \vee \bigvee B_j(x))$
7	$Disj(s_1, s_2)$	$\forall xy.(s_1(x,y) \wedge s_2(x,y) \rightarrow \bot)$
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THEOREM (UPPER COMPLEXITY FOR $\mathcal{SROIQ}$)

(Finite model) reasoning in $\mathcal{SROIQ}$ is N2ExpTime

Outline

Summary

- New complexity results:
  - $\mathcal{SROIQ}$ and $\mathcal{SROIF}$ are N2ExpTime;
  - $\mathcal{SRIQ}$, $\mathcal{RIQ}$, and $\mathcal{R}$ are 2ExpTime-hard.
- Complexity blowup is caused by complex RIAs:
  - either by $S_1 \circ \cdots \circ S_n \sqsubseteq R$,
  - or by $R \circ S \sqsubseteq R \quad + \quad S \circ R \sqsubseteq R$
- Explains why the exponential blowup in the tableau procedures for $\mathcal{SRIQ}$ and $\mathcal{SROIQ}$ is unavoidable

SUMMARY

- New complexity results:
  - $\mathcal{SROIQ}$ and $\mathcal{SROIF}$ are N2ExpTime;
  - $\mathcal{SRIQ}$, $\mathcal{RIQ}$, and $\mathcal{R}$ are 2ExpTime-hard.
- Complexity blowup is caused by complex RIAs:
  - either by $S_1 \circ \cdots \circ S_n \sqsubseteq R$,
  - or by $R \circ S \sqsubseteq R$ + $S \circ R \sqsubseteq R$
- Explains why the exponential blowup in the tableau procedures for $\mathcal{SRIQ}$ and $\mathcal{SROIQ}$ is unavoidable
- Open questions:
  1. Upper bound for $\mathcal{SRIQ}$ & $\mathcal{RIQ}$? Conjecture: 2ExpTime
  2. Complexity of $\mathcal{RIQ}$ with only left-linear / right-linear axioms?

Avoiding the Exponential Blowup

- Some further restrictions on complex RIAs are known to prevent an exponential blowup

  (e.g. when every sequence $R_1 \prec R_2 \prec \cdots \prec R_n$ has a bounded length)

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- Only the size of the **RBox** has a high complexity impact:

$\mathcal{SH}[\mathcal{O}]\mathcal{IQ}$

ABox	TBox	RBox
NP ?		
[N]ExpTime		
[N]ExpTime		

$\mathcal{SR}[\mathcal{O}]\mathcal{IQ}$

ABox	TBox	RBox
NP ?		
[N]ExpTime		
2[N]ExpTime		

Questions?

- Please send difficult questions to

Yevgeny Kazakov
yevgeny.kazakov@comlab.ox.ac.uk

- Our contribution:
  1. $\mathcal{SROIQ}$ and $\mathcal{SROIF}$ are N2ExpTime-complete
  2. $\mathcal{SRIQ}$, $\mathcal{RIQ}$, and $\mathcal{R}$ are 2ExpTime-hard

- Thank you for your attention!