A Note on the Complexity of the Satisfiability Problem for Graded Modal Logics

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***Abstract*—*Graded modal logic* is the formal language obtained from ordinary modal logic by endowing its modal operators with cardinality constraints. Under the familiar possible-worlds semantics, these augmented modal operators receive interpretations such as "It is true at no fewer than 15 accessible worlds that ...", or "It is true at no more than 2 accessible worlds that ...". We investigate the complexity of satisfiability for this language over some familiar classes of frames. This problem is more challenging than its ordinary modal logic counterpart—especially in the case of transitive frames, where graded modal logic lacks the tree-model property. We obtain tight complexity bounds for the problem of determining the satisfiability of a given graded modal logic formula over the classes of frames characterized by any combination of reflexivity, seriality, symmetry, transitivity and the Euclidean property.**

***Keywords*-modal logic; graded modalities; computational complexity**

I. Introduction

Graded modal logic is the formal language obtained by decorating the $\Diamond$-operator of ordinary modal logic with subscripts expressing cardinality constraints. Specifically, for $C \geq 0$, the formula $\Diamond_{\leq C}\varphi$ may be glossed: "φ is true at no more than C accessible worlds," and the formula $\Diamond_{\geq C}\varphi$ may be glossed: "φ is true at no fewer than C accessible worlds." The semantics for graded modal logic generalize the relational semantics for ordinary modal logic in the expected way. We employ the labels Rfl, Ser, Sym, Tr and Eucl to denote, respectively, the classes of reflexive, serial, symmetric, transitive and Euclidean frames. (Definitions of these frame classes are given in Table I.) Using this notation, $\bigcap\{\text{Rfl}, \text{Tr}\}$ denotes the class of reflexive, transitive frames, $\bigcap\{\text{Ser}, \text{Tr}, \text{Eucl}\}$ denotes the class of serial, transitive, Euclidean frames, and so on. As a limiting case, $\bigcap\emptyset$ denotes the class of all frames. In this paper, we investigate the computational complexity of determining the satisfiability of a given formula of graded modal logic over any frame class of the form $\bigcap\mathcal{F}$, where $\mathcal{F} \subseteq \{\text{Rfl}, \text{Ser}, \text{Sym}, \text{Tr}, \text{Eucl}\}$.

It is easy to see that ordinary modal logic is in effect a sub-language of graded modal logic: any formula of the form $\Diamond\varphi$ may be equivalently written $\Diamond_{\geq 1}\varphi$, and similarly, any formula of the form $\Box\varphi$ may be equivalently written $\Diamond_{\leq 0}\neg\varphi$. And ordinary modal logic provides a good starting point for our analysis, because its complexity-theoretic treatment is comparatively straightforward. The following two theorems are well-known, and may be proved using techniques found in any modern text on modal logic (e.g. [1]). We remind the reader that symmetry and transitivity together imply the Euclidean property.

Theorem 1. *Let* $\mathcal{F} \subseteq \{\text{Rfl}, \text{Ser}, \text{Sym}, \text{Tr}, \text{Eucl}\}$*, with* $\text{Eucl} \in \mathcal{F}$ *or* $\{\text{Sym}, \text{Tr}\} \subseteq \mathcal{F}$*. Then the satisfiability problem for ordinary modal logic over* $\bigcap\mathcal{F}$ *is* NP*-complete.*

Theorem 2. *If* $\mathcal{F} \subseteq \{\text{Rfl}, \text{Ser}, \text{Tr}\}$*, then the satisfiability problem for ordinary modal logic over* $\bigcap\mathcal{F}$ *is* PSpace*-complete* [2]*. Also, if* $\mathcal{F} \subseteq \{\text{Rfl}, \text{Ser}, \text{Sym}\}$*, then the satisfiability problem for ordinary modal logic over* $\bigcap\mathcal{F}$ *is* PSpace*-complete.*

The upper complexity bound in Theorem 1 follows from the fact that ordinary modal logic has the polynomial-size model property over the relevant frame classes: if a formula φ of ordinary modal logic is satisfiable over a frame in $\bigcap\mathcal{F}$, where $\mathcal{F}$ satisfies the conditions of Theorem 1, then it is satisfiable over a frame in $\bigcap\mathcal{F}$ whose size is bounded by a polynomial function of the number of symbols in φ. For the frame classes of Theorem 2, ordinary modal logic lacks the polynomial-size model property. However, it does have the *tree-model* property: if a formula is satisfiable over a frame in any of the classes $\bigcap\mathcal{F}$ mentioned in Theorem 2, then it is satisfiable over a frame in that class which forms a (possibly infinite) tree [3]. Because the branches of this tree can be assumed to be either short or periodic with small period, and because these branches can be explored one-by-one, the PSpace-upper complexity bound may be obtained by exhibiting, for each relevant frame class $\bigcap\mathcal{F}$, a suitable semantic tableau algorithm.

Turning our attention to the language of graded modal logic, our first question is whether the results of Theorems 1 and 2 carry over to the larger language. When $\mathcal{F}$ contains neither of the classes Tr or Eucl, the answer is yes. We have:

Theorem 3. *The satisfiability problem for graded modal logic over* $\mathcal{F} = \bigcap\emptyset$ *is* PSpace*-complete* [4]*. In fact, if* $\mathcal{F} \subseteq \{\text{Rfl}, \text{Ser}, \text{Sym}\}$*, then the satisfiability problem for graded modal logic over* $\bigcap\mathcal{F}$ *is* PSpace*-complete.*

The reason—and indeed the reasoning—is essentially the same as for Theorem 2: the PSpace upper complexity bound in Theorem 3 depends on the fact that graded modal logic enjoys the tree-model property over the relevant frame classes. This can then be used to establish the correctness of semantic tableau algorithms for graded modal logic over these frame classes. The paper [4] actually considers only the case $\mathcal{F} = \emptyset$ (i.e. the class of all frames); however, the modifications required to take account of reflexivity, seriality and symmetry are routine, because these restrictions do not compromise the tree-model property. Note that the upper complexity bound in Theorem 3 holds even when numerical subscripts are coded in binary. (The much easier result for unary coding can be found in [5].)

When $\mathcal{F}$ contains either Eucl or Tr, the complexity of the satisfiability problem for graded modal logic over $\bigcap \mathcal{F}$ is harder to determine. Consider first the analogue of Theorem 1, where we have either $\text{Eucl} \in \mathcal{F}$ or $\{\text{Tr}, \text{Sym}\} \subseteq \mathcal{F}$, and let $\{\varphi_n\}_{n \geq 0}$ be the sequence of formulas given by $\varphi_n = \Diamond_{\geq 2^n} p$. Assuming binary coding of numerical subscripts, the number of symbols in φ_n is bounded by a linear function of n, and every φ_n is satisfiable over a Euclidean frame; but φ_n is certainly not satisfiable over any frame with fewer than 2^n worlds! Thus, for graded modal logic, the reasoning used to prove Theorem 1 fails. Nevertheless, the corresponding complexity result still holds:

Theorem 4. *Let $\mathcal{F} \subseteq \{\text{Rfl}, \text{Ser}, \text{Sym}, \text{Tr}, \text{Eucl}\}$, with $\text{Eucl} \in \mathcal{F}$ or $\{\text{Sym}, \text{Tr}\} \subseteq \mathcal{F}$. Then the satisfiability problem for graded modal logic over $\bigcap \mathcal{F}$ is* NP*-complete.*

We prove Theorem 4 in Section III.

When $\mathcal{F}$ contains Tr, but neither Sym nor Eucl, we cannot apply the reasoning of Theorem 2 at all, since graded modal logic lacks the tree-model property over transitive frames. For example, consider the formula φ given by

$$\varphi := q_0 \wedge \Diamond_{\geq 2}(\neg q_0 \wedge q_1 \wedge \Diamond_{\geq 1}(\neg q_0 \wedge \neg q_1)) \wedge \Diamond_{\leq 1} \neg q_1.$$

The formula φ is certainly satisfiable over transitive frames; however, it is not satisfiable over tree-shaped transitive frames. For suppose φ is true at a world w_0 in some structure. The conjunct $\Diamond_{\geq 2}(\neg q_0 \wedge q_1 \wedge \Diamond_{\geq 1}(\neg q_0 \wedge \neg q_1))$ ensures the existence of distinct worlds w_1 and w_2, accessible from (and distinct from) w_0, and, for $i = 1, 2$, a world w_i' accessible from w_i and satisfying $\neg q_1$, with w_i' distinct from w_0, w_1 and w_2. But the conjunct $\Diamond_{\leq 1} \neg q_1$ ensures that, if the accessibility relation is transitive, $w_1' = w_2'$. Hence, φ is not satisfiable over a tree. Indeed, we show below that, for the relevant frame classes, graded modal logic and ordinary modal logic exhibit different complexities:

Theorem 5. *Let $\mathcal{F} \subseteq \{\text{Rfl}, \text{Ser}, \text{Tr}\}$, with $\text{Tr} \in \mathcal{F}$. Then the satisfiability problem for graded modal logic over $\bigcap \mathcal{F}$ is* NExpTime*-complete. It remains* NExpTime*-hard, even when all numerical subscripts in modal operators are at most* 1.

We prove Theorem 5 in Section IV. The final statement of the theorem is significant, because it means that the result does not depend upon the coding of numerical subscripts.

A moment's thought shows that the conditions in Theorems 3–5 are exhaustive: together, they establish the complexity of the satisfiability problem for graded modal logic over $\bigcap \mathcal{F}$ for every $\mathcal{F} \subseteq \{\text{Rfl}, \text{Ser}, \text{Sym}, \text{Tr}, \text{Eucl}\}$.

The *decidability* of the satisfiability problem for graded modal logic over various frame classes $\bigcap \mathcal{F}$ is touched on in [6], where it is stated (p. 520) that "standard techniques or modifications of them may be used to prove the decidability of most of [these] logics"; however, the paper gives no further details. Several such decidability results are claimed in [7]; however, in the (difficult) case where $\mathcal{F} = \{\text{Tr}\}$, this proof contains an error, as reported in [8]. The latter provides a correct proof; however, the method employed there does not establish any complexity bounds. It is conjectured in [9] (Remark 4.12), that the satisfiability problem for graded modal logic over the class of transitive, symmetric and reflexive frames is PSpace-complete: Theorem 4 shows that this conjecture, if true, would imply that PSpace=NP. Earlier accounts of graded modal logics focused primarily on the problem of axiomatizing the set of valid formulas over these frame classes. For instance, [6] provides (or reports) such axiomatizations for $\bigcap \mathcal{F}$, where $\mathcal{F}$ is any of $\emptyset$, $\{\text{Rfl}\}$, $\{\text{Sym}\}$, $\{\text{Rfl}, \text{Sym}\}$, $\{\text{Rfl}, \text{Tr}\}$ and $\{\text{Rfl}, \text{Tr}, \text{Sym}\}$. Similar results can be found in [10], [11], [12], [13]; see also [9] for axiomatizations of some related logics.

Graded modal logics are closely related to terminological languages and description logics (DLs) [14] featuring so-called qualified number restrictions. These logics allow concepts to be defined by specifying how many things (of various kinds) instances of those concepts can be related to. Logics featuring both qualified number restrictions and transitive relations are frequently undecidable [15], and many DLs incorporate various syntactic restrictions to restore decidability. It was recently shown in [8] that some of these syntactic restrictions can be considerably relaxed.

Because of space limitations, we provide only an outline of some proofs, or omit them altogether. Full details can be found in [16].

II. Preliminaries

Fix a countably infinite set Π. The language of *graded modal logic* is defined to be the smallest set of expressions, $\mathcal{GM}$, satisfying the following conditions:

1) $\Pi \subseteq \mathcal{GM}$;
2) if φ and ψ are in $\mathcal{GM}$, then so are $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$ and $\varphi \leftrightarrow \psi$;
3) if φ is in $\mathcal{GM}$, then so are $\Diamond_{\leq C}\varphi$ and $\Diamond_{\geq C}\varphi$, for any bit-string C.

We refer to expressions in this set as *$\mathcal{GM}$-formulas* (or simply *formulas*, if clear from context). If φ is a $\mathcal{GM}$-formula, we take the *size of* φ, denoted $\|\varphi\|$, to be the number of symbols in φ. Throughout the paper, we equivocate between bit-strings and the natural numbers they represent in the usual way. Thus, we may informally think of the subscripts in $\Diamond_{\leq C}$ and $\Diamond_{\geq C}$ as natural numbers, it being understood that the number of symbols in, for example, $\Diamond_{\leq C}$ is approximately $\log C$, rather than C. That is: in giving the size of a formula, we assume *binary*, rather than *unary*, coding.

Let Σ be the relational signature with unary predicates Π and single binary predicate r, and let $\mathfrak{A}$ be a Σ-structure with domain W. We refer to the elements of W as *worlds*. We define the *satisfaction* relation for $\mathcal{GM}$-formulas inductively as follows:

1) $\mathfrak{A} \models_w p$ if and only if $w \in p^{\mathfrak{A}}$;
2) $\mathfrak{A} \models_w \neg\varphi$ if and only if $\mathfrak{A} \not\models_w \varphi$, and similarly for $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$;
3) $\mathfrak{A} \models_w \Diamond_{\geq C}\varphi$ if and only if there exist at least C worlds $v \in W$ such that $\langle w, v\rangle \in r^{\mathfrak{A}}$ and $\mathfrak{A} \models_v \varphi$;
4) $\mathfrak{A} \models_w \Diamond_{\leq C}\varphi$ if and only if there exist at most C worlds $v \in W$ such that $\langle w, v\rangle \in r^{\mathfrak{A}}$ and $\mathfrak{A} \models_v \varphi$.

The notion of satisfaction extends to sets of $\mathcal{GM}$-formulas Φ as expected: $\mathfrak{A} \models_w \Phi$ if $\mathfrak{A} \models_w \varphi$ for all $\varphi \in \Phi$. If $\mathfrak{A} \models_w \varphi$, we sometimes say, informally, that φ is *true at* w *in* $\mathfrak{A}$. We write $\Box\varphi$ as an abbreviation for $\Diamond_{\leq 0}\neg\varphi$, and $\Diamond\varphi$ as an abbreviation for $\Diamond_{\geq 1}\varphi$, or, equivalently, $\neg\Diamond_{\leq 0}\varphi$. Thus, the language of ordinary modal logic may be regarded as the subset of $\mathcal{GM}$ in which all indices are restricted to 0. Finally, we write $\boxdot\varphi$ as an abbreviation for $\varphi \wedge \Box\varphi$.

By a *frame*, we mean an $\{r\}$-structure—in other words, a non-empty (possibly infinite) digraph. If $\mathfrak{A}$ is a Σ-structure, then its $\{r\}$-reduct is a frame $\mathfrak{F}$: we say that $\mathfrak{A}$ is a structure *over* $\mathfrak{F}$. Further, we call the mapping $V : \Pi \rightarrow \mathbb{P}(W)$ given by $p \mapsto p^{\mathfrak{A}}$ the *valuation* of $\mathfrak{A}$ (*on* W). We write $\mathfrak{A} = (W, R, V)$ to indicate that $\mathfrak{A}$ is a Σ-structure over the frame (W, R) with valuation V. Obviously, this determines $\mathfrak{A}$ completely. Henceforth, the term "structure", with no signature qualification, will always mean "Σ-structure". Let φ be a $\mathcal{GM}$-formula. We say that φ is *satisfiable over* a frame $\mathfrak{F}$ if there exists a structure $\mathfrak{A}$ over $\mathfrak{F}$ and a world w of $\mathfrak{A}$ such that $\mathfrak{A} \models_w \varphi$. Further, φ is *satisfiable over* a class of frames $\mathcal{K}$ if it is satisfiable over some frame in $\mathcal{K}$. We denote by $\mathcal{GM}_{\mathcal{K}}$*-Sat* the problem of determining whether a given $\mathcal{GM}$-formula is satisfiable over $\mathcal{K}$.

Any first-order sentence α over the signature $\{r\}$ defines a class of frames $\{\mathfrak{F} : \mathfrak{F} \models \alpha\}$. The most common frame classes are those which we agreed in Section I to denote by the labels Rfl, Ser, Sym, Tr and Eucl. Table I lists these frame classes together with their respective defining first-order sentences. A structure over a reflexive frame will simply be called a *reflexive* structure, and similarly for the other frame properties. We can now articulate the objective of this paper. Let $\mathcal{F}$ be a subset (possibly empty) of the set of frame classes $\{\text{Rfl}, \text{Ser}, \text{Sym}, \text{Tr}, \text{Eucl}\}$. We ask: what is the complexity of $\mathcal{GM}_{\cap\mathcal{F}}$-Sat?

Table I
FRAME CLASSES CONSIDERED IN THIS PAPER.

reflexive frames	$\forall x.r(x,x)$
serial frames	$\forall x \exists y.r(x,y)$
symmetric frames	$\forall x \forall y.(r(x,y) \rightarrow r(y,x))$
transitive frames	$\forall x \forall y \forall z.(r(x,y) \wedge r(y,z) \rightarrow r(x,z))$
Euclidean frames	$\forall x \forall y \forall z.(r(x,y) \wedge r(x,z) \rightarrow r(y,z))$.

III. Euclidean frames

The purpose of this section is to prove Theorem 4. We make use of a known complexity result on first-order logic with counting quantifiers. Denote by $\mathcal{C}^1$ the set of first-order formulas featuring only a single variable x, but with the counting quantifiers $\exists_{\leq C} x$ and $\exists_{\geq C} x$ allowed. The following result holds for both unary and binary coding of numerical subscripts:

Theorem 6 ([17], [18])**.** *The problem of deciding satisfiability for $\mathcal{C}^1$-formulas is* NP*-complete.*

We show that, for $\mathcal{GM}$-formulas, satisfiability over Euclidean frames is equivalent to satisfiability over frames having a particularly simple form, and that, for such frames, the fragment $\mathcal{C}^1$ is as expressive as we need.

Let $\mathfrak{F} = (W, R)$ be a frame. If $X \subseteq W$, $R(X)$ denotes $\bigcup_{x \in X}\{w \in W \mid \langle x, w\rangle \in R\}$; we write $R(w)$ for $R(\{w\})$. If $\mathfrak{F} = (W, R)$ is a frame, and $X \subseteq W$, $R^*(X)$ denotes $X \cup R(X) \cup R(R(X)) \cup \cdots$; we write $R^*(w)$ for $R^*(\{w\})$. If $\mathfrak{A}$ is a structure over a frame (W, R) and $X \subseteq W$, let $\mathfrak{B}$ be the substructure of $\mathfrak{A}$ with domain $R^*(X)$. We call $\mathfrak{B}$ the *substructure* generated by X. Note that reflexivity, seriality, symmetry, transitivity and the Euclidean property are all preserved under generated substructures.

Lemma 1. *Let φ be a formula of $\mathcal{GM}$, $\mathfrak{A}$ a structure, w a world of $\mathfrak{A}$ and $\mathfrak{B}$ the substructure generated by $\{w\}$. If $\mathfrak{A} \models_w \varphi$, then $\mathfrak{B} \models_w \varphi$.*

Proof: Induction on the structure of φ. ■

Lemma 2. *Let $\mathfrak{F} = (W, R)$ be a Euclidean frame and $w_0 \in W$. Then:* (*i*) *$R(w_0) \subseteq R(R(w_0))$,* (*ii*) *$R^*(w_0) = \{w_0\} \cup R(R(w_0))$, and* (*iii*) *$R$ is total on $R(R(w_0))$.*

Proof: Routine. ■

Lemmas 1 and 2 show that, when discussing satisfiability over Euclidean frames, we may restrict attention to frames of the form $(W \cup \{w_0\}, R)$, where R is total on W, $R(w_0) \subseteq W$, and w_0 may or may not be in W. Over such simple frames, any $\mathcal{GM}$-formula can be translated into an equisatisfiable $\mathcal{C}^1$-formula. Specifically:

Lemma 3. *Let $\mathcal{F} \subseteq \{\text{Rfl}, \text{Ser}, \text{Sym}, \text{Tr}\}$. Given a $\mathcal{GM}$-formula φ, we can compute, in time bounded by a polynomial function of $\|\varphi\|$, a $\mathcal{C}^1$-formula α such that φ is satisfiable over a frame in $\bigcap\mathcal{F} \cap$* Eucl *if and only if α is satisfiable.*

Proof: Omitted. ∎

The upper bound of Theorem 4 now follows by Theorem 6 and Lemma 3, since $\mathrm{Sym} \cap \mathrm{Tr} \subseteq \mathrm{Eucl}$. The lower bound is trivial, since $\mathcal{GM}$ includes propositional logic.

IV. Transitive frames

The purpose of this section is to establish Theorem 5. The upper bound (Section IV-A) is obtained by proving that every $\mathcal{GM}$-formula φ that is satisfiable over a transitive (transitive and reflexive) frame is also satisfiable over a transitive (transitive and reflexive) frame whose size is bounded by an exponential function of $\|\varphi\|$. It is shown in [8] that every $\mathcal{GM}$-formula satisfiable over a transitive frame is also satisfiable over a *finite* transitive frame. However, this paper gives no bound on the size of the satisfying structure. The matching lower bound (Section IV-B) is obtained by a reduction from exponential tiling problems. Interestingly, this reduction features only formulas in which all numerical subscripts are bounded by 1. Thus, the lower complexity-bound of Theorem 5 continues to hold even under unary coding of numerical subscripts.

One note on terminology before we proceed. In the context of (graded) modal logic, it is customary to think of the unary predicates in Π as *proposition letters*, because they receive truth-values relative to worlds. Since we shall not be concerned with $\mathcal{C}^1$ or other first-order fragments in the sequel, we adopt this practice from now on. Accordingly, a *propositional* formula is one containing no modal operators. Finally, we shall relax our stance on valuations, allowing structures to interpret only those proposition letters involved in some collection of formulas of interest, rather than every proposition letter in Π.

A. Membership in NExpTime

First we demonstrate that every $\mathcal{GM}$-formula can be transformed into a normal form preserving satisfiability over transitive frames. This normal form is broadly similar to the so-called Scott normal form for the two-variable fragment of first-order logic, and is likewise obtained by a straightforward renaming procedure. For the next lemma, recall that $\boxdot\varphi$ abbreviates $\varphi \wedge \Box\varphi$.

Lemma 4. *Let φ be a $\mathcal{GM}$-formula. We can compute, in time bounded by a polynomial function of $\|\varphi\|$, a $\mathcal{GM}$-formula ψ of the form*

$$\eta \wedge \boxdot\big(\theta \wedge \bigwedge_{1\le i\le \ell} (p_i \to \Diamond_{\ge C_i}\pi_i) \wedge \bigwedge_{1\le j\le m} (q_j \to \Diamond_{\le D_j}\chi_j)\big), \quad (1)$$

where the p_i and the q_j are proposition letters, the C_i and D_j are natural numbers, and η, θ, the π_i and the χ_j are propositional formulas, such that φ and ψ are satisfiable over exactly the same transitive frames.

Proof: Suppose that $\Diamond_{\le D}\pi$ is a sub-formula of φ, with π propositional. Let φ' be the formula

$$\varphi[p] \wedge \boxdot((p \vee q) \wedge (p \to \Diamond_{\le D}\pi) \wedge (q \to \Diamond_{\ge D+1}\pi)),$$

where p and q are new elements of Π (not occurring in φ), and $\varphi[p]$ denotes the result of substituting p for every occurrence of $\Diamond_{\le D}\pi$ in φ. Then φ and φ' are satisfiable over the same transitive frames. By repeating this process (treating subformulas of the form $\Diamond_{\ge D}\pi$ similarly), we eventually obtain the desired formula ψ. ∎

We next present lemmas describing transformations of transitive structures, in which we use the following terminology. Let $\mathfrak{A} = \langle W, R, V\rangle$ be a transitive structure, and w_1, w_2 be worlds of W. We say: w_2 is an *R-successor of* w_1 if $\langle w_1, w_2\rangle \in R$; w_2 is a *strict R-successor of* w_1 if $\langle w_1, w_2\rangle \in R$, but $\langle w_2, w_1\rangle \notin R$; w_1 and w_2 are *R-equivalent* if $\langle w_1, w_2\rangle \in R$ and $\langle w_2, w_1\rangle \in R$. The *$R$-clique for* w_1 in $\mathfrak{A}$ is the set $Q_{\mathfrak{A}}(w_1) \subseteq W$ consisting of w_1 and all worlds R-equivalent to w_1. We say that w_2 is a *direct R-successor of* w_1 if w_2 is a strict R-successor of w_1 and, for every $w \in W$ such that $\langle w_1, w\rangle \in R$ and $\langle w, w_2\rangle \in R$, we have either $w \in Q_{\mathfrak{A}}(w_1)$ or $w \in Q_{\mathfrak{A}}(w_2)$.

The *depth of a structure* $\mathfrak{A}$ is the maximum over all $k \ge 0$ for which there exist worlds $w_0, \dots, w_k \in W$ such that w_i is a strict R-successor of w_{i-1} for every i with $1 \le i \le k$, or ∞ if no such maximum exists. The *breadth of* $\mathfrak{A}$ is the maximum over all $k \ge 0$ for which there exist worlds $w, w_1, \dots, w_k$ such that w_i is a direct R-successor of w for every i with $1 \le i \le k$, and the sets $Q_{\mathfrak{A}}(w_1), \dots, Q_{\mathfrak{A}}(w_k)$ are disjoint, or ∞ if no such maximum exists. The *width of* $\mathfrak{A}$ is the smallest k such that $k \ge \|Q_{\mathfrak{A}}(w)\|$ for all $w \in W$, or ∞ if no such k exists.

Lemma 5. *Let $\mathfrak{A}$ be a structure of depth d, breadth b and width c (all finite), and let w be a world of $\mathfrak{A}$. Then the substructure of $\mathfrak{A}$ generated by $\{w\}$ contains no more than n worlds, where $n = c$ if $b = 0$, $n = c \cdot (d+1)$ if $b = 1$, and $n = c \cdot (b^{d+1} - 1)/(b-1)$ otherwise.*

Proof: Elementary. ∎

We employ the following notation. For a structure $\mathfrak{A} = (W, R, V)$ and a binary relation R' on W (possibly different from R), we denote by $R'_{\mathfrak{A}}(w, \varphi)$ the set $\{v \mid \langle w, v\rangle \in R', \mathfrak{A} \models_v \varphi\}$. Thus, $\mathfrak{A} \models_w \Diamond_{\ge C}\varphi$ if and only if $\|R_{\mathfrak{A}}(w, \varphi)\| \ge C$, where $\|S\|$ denotes the cardinality of the set S. Similarly, $\mathfrak{A} \models_w \Diamond_{\le C}\varphi$ if and only if $\|R_{\mathfrak{A}}(w, \varphi)\| \le C$.

Lemma 6. *Let φ be a formula of the form* (1). *If φ has a transitive model $\mathfrak{A}$, then it has a transitive model $\mathfrak{A}'$ with depth $d' \le 2\ell$, breadth $b' \le \sum_{i=1}^{\ell} C_i$ and width $c' \le \sum_{i=1}^{\ell} C_i + 1$. If $\mathfrak{A}$ is reflexive, then we can additionally ensure that $\mathfrak{A}'$ is also reflexive.*

Proof sketch: Let $\mathfrak{A} = (W, R, V)$. We construct $\mathfrak{A}' = (W', R', V')$ from $\mathfrak{A}$ in four stages.

Stage 1: Adapting a technique employed in [8] to establish the finite model property for $\mathcal{GM}$-formulas, we first define a transitive model $\mathfrak{A}'$ of φ, reflexive if $\mathfrak{A}$ is, such that $\mathfrak{A}'$ has finite depth. The strategy is to *enlarge* the relation R (thus reducing the number of *strict* successors of worlds in W), preserving satisfaction for subformulas of the form $\Diamond_{\leq D_j}\chi_j$.

For $w \in W$ define $d^j_{\mathfrak{A}}(w) := \min(D_j+1, \|R^*(w,\chi_j)\|)$ where D_j and χ_j $(1 \leq j \leq m)$ are as in (1), and R^* is the reflexive closure of R. Let $R_d := \{\langle w_1, w_2\rangle \in R \mid d^j_{\mathfrak{A}}(w_1) = d^j_{\mathfrak{A}}(w_2),\ 1 \leq j \leq m\}$ be the restriction of R to pairs of elements that have the same values of $d^j_{\mathfrak{A}}(w)$, and let $R^-_d := \{\langle w_1, w_2\rangle \mid \langle w_2, w_1\rangle \in R_d\}$ be the inverse of R_d. Let $\mathfrak{A}' = (W, R', V)$ be obtained from $\mathfrak{A} = (W, R, V)$ by setting $R' := (R \cup R^-_d)^+$. It can now be shown that $\mathfrak{A}'$ satisfies φ, is reflexive if $\mathfrak{A}$ is, and has finite depth.

Stage 2: By Stage 1, we may assume that $\mathfrak{A}$ has finite depth d. We define a transitive model $\mathfrak{A}'$ of φ, reflexive if $\mathfrak{A}$ is, such that $\mathfrak{A}'$ has depth $d' \leq 2\ell$. If $d \leq 2\ell$ then we take $\mathfrak{A}' = \mathfrak{A}$. Otherwise, we obtain $\mathfrak{A}'$ from $\mathfrak{A}$ by *contracting* the relation R (removing unnecessary *direct* successors of worlds in W), preserving satisfaction for subformulas of the form $\Diamond_{\geq C_i}\pi_i$. Define, for every $w \in W$, two sets of indices:

$$I_{\mathfrak{A}}(w) = \{i \mid 1 \leq i \leq \ell, \|R(w,\pi_i)\| \geq C_i\}, \text{ and}$$
$$I^s_{\mathfrak{A}}(w) = \{i \mid 1 \leq i \leq \ell, \|R(w,\pi_i) \setminus Q_{\mathfrak{A}}(w)\| \geq C_i\},$$

where π_i and C_i are as in (1), $1 \leq i \leq \ell$. Define the structure $\mathfrak{A}' = \langle W, R', V\rangle$ by setting

$$R' := R \setminus \{\langle w_1, w_2\rangle \mid w_2 \text{ is a direct } R'\text{-successor of } w_1 \text{ and } I^s_{\mathfrak{A}}(w_2) = I_{\mathfrak{A}}(w_1)\}.$$

It can be shown that $\mathfrak{A}'$ is a transitive structure which satisfies φ, is reflexive if $\mathfrak{A}$ is, and has depth $d' < d$. Repeating this step sufficiently often, we eventually ensure that $d' \leq 2\ell$.

Stage 3: By Stage 2, we may assume that $\mathfrak{A}$ has depth $d \leq 2\ell$. We define a transitive model $\mathfrak{A}'$ of φ, reflexive if $\mathfrak{A}$ is, such that $\mathfrak{A}'$ has depth $d' \leq 2\ell$ and breadth $b' \leq \sum_{i=1}^{\ell} C_i$. For every element $w \in W$ and every i with $1 \leq i \leq \ell$, let $W_i(w)$ be the set of strict R-successors of w for which π_i holds. We call the elements of $W_i(w)$ the *strict π_i-witnesses for w*. Note that $W_i(w_1) = W_i(w_2)$ when w_1 and w_2 are R-equivalent. Let $W'_i(w)$ be $W_i(w)$ if $\|W_i(w)\| \leq C_i$ or, otherwise, a subset of $W_i(w)$ which contains exactly C_i elements. We call $W'_i(w)$ the *selected strict π_i-witnesses for w*. We assume that $W'_i(w_1) = W'_i(w_2)$ when w_1 and w_2 are R-equivalent. Let $R_q := \{\langle w, w'\rangle \in R \mid w' \in Q_{\mathfrak{A}}(w)\}$ be the restriction of R to elements of the same clique, and $R'_i = \{\langle w, w'\rangle \in R \mid w' \in W'_i(w)\}$ be the relation between an element $w \in W$ and the selected strict π_i-witnesses for w. Define the structure $\mathfrak{A}' = (W, R', V)$ by setting $R' := (R_q \cup \bigcup_{1 \leq i \leq \ell} R'_i)^+$. Intuitively, $\mathfrak{A}'$ is obtained from $\mathfrak{A}$ by removing all strict successor relations except those that are induced by selected strict witnesses. It may then be shown that $\mathfrak{A}'$ has the required properties.

Stage 4: By Stage 3, we may assume that $\mathfrak{A}$ has depth $d \leq 2\ell$ and breadth $b \leq \sum_{i=1}^{\ell} C_i$. We define a structure $\mathfrak{A}'$ with all the properties required by the lemma. For every element $w \in W$, and every i with $1 \leq i \leq \ell$, let $Q_i(w)$ be the set of elements in $Q_{\mathfrak{A}}(w)$ for which π_i holds. We call the elements of $Q_i(w)$ the *equivalent π_i-witnesses for w*. Note that $Q_i(w_1) = Q_i(w_2)$ when w_1 and w_2 are R-equivalent. Let $Q'_i(w)$ be $Q_i(w)$ if $\|Q_i(w)\| \leq C_i$ or, otherwise, a subset of $Q_i(w)$ which contains exactly C_i elements. We call $Q'_i(w)$ the *selected equivalent π_i-witnesses for w*. Also let $Q'_0(w)$ be a singleton set containing an element of $Q_{\mathfrak{A}}(w)$ that satisfies φ if there is one, and any element of $Q_{\mathfrak{A}}(w)$ otherwise. We assume that $Q'_i(w_1) = Q'_i(w_2)$ when w_1 and w_2 are R-equivalent. Define the structure $\mathfrak{A}' = \langle W', R', V'\rangle$ by setting $W' := \bigcup_{w \in W,\, 0 \leq i \leq \ell} Q'_i(w)$, $R' := R|_{W'}$, and $V' := V|_{W'}$. Intuitively $\mathfrak{A}'$ is obtained from $\mathfrak{A}$ by removing elements in every R-clique, except for those that are selected witnesses of other elements, and in such a way that the clique remains non-empty and contains at least one element satisfying φ if there was one. It may then be shown that $\mathfrak{A}'$ has the required properties. ∎

Lemma 7. *Let $\mathfrak{A} = \langle W, R, V\rangle$ be a transitive structure that satisfies a formula φ of the form* (1)*. Then there exists a transitive structure $\mathfrak{A}' = \langle W', R', V'\rangle$ that satisfies φ such that $\|W'\| \leq (b+1)\cdot(b^{2\ell+1}-1)/(b-1)$, where $b = \max(2, \sum_{i=1}^{\ell} C_i)$. Moreover, if $\mathfrak{A}$ is reflexive, then we can ensure that $\mathfrak{A}'$ is also reflexive.*

Proof: By Lemma 6, there is a transitive structure $\mathfrak{A}'$ satisfying φ, reflexive if $\mathfrak{A}$ is, with depth, breadth, and width bounded respectively by 2ℓ, b, and $b+1$. Let w_0 be such that $\mathfrak{A}' \models_{w_0} \varphi$, and consider the substructure of $\mathfrak{A}'$ generated by $\{w_0\}$. The result now follows by Lemmas 1 and 5. ∎

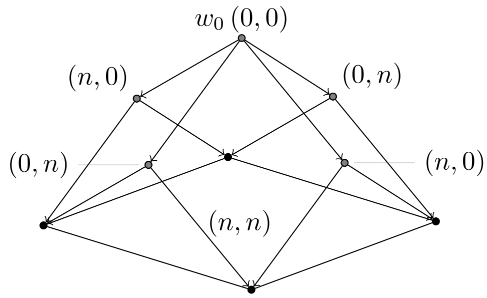
We remark that the bound $(b+1)\cdot(b^{2\ell+1}-1)/(b-1)$ obtained in Lemma 7 is at most exponential in the size of the input formula, even under binary coding of the numerical subscripts $C_1, \ldots, C_\ell$. Notice, incidentally, that this bound does not mention the subscripts $D_1, \ldots, D_m$ at all.

Corollary 1. *If $\mathcal{F}$ is any of* {Tr}, {Rfl, Tr} *or* {Ser, Tr}*, then the problem $\mathcal{GM}_{\cap\mathcal{F}}$-Sat is in* NExpTime.

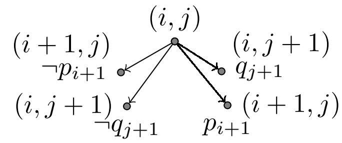
Proof: Consider first the cases $\mathcal{F} = \{\text{Tr}\}$ and $\mathcal{F} = \{\text{Tr}, \text{Rfl}\}$. By Lemma 4, any $\mathcal{GM}$ formula φ can be transformed in polynomial time into a formula ψ of the form (1) preserving satisfiability over $\bigcap\mathcal{F}$. By Lemma 7, ψ is satisfiable over $\bigcap\mathcal{F}$ if and only if it is satisfiable over a frame in $\bigcap\mathcal{F}$ of size at most exponential in $\|\psi\|$. This last condition can be checked in non-deterministic exponential time. Finally, using Lemma 1, a formula φ is satisfiable over Ser$\cap$Tr if and only if $\varphi \wedge \boxdot\Diamond\top$ is satisfiable over Tr, where $\top$ is any tautology. ∎

B. NExpTime*-hardness*

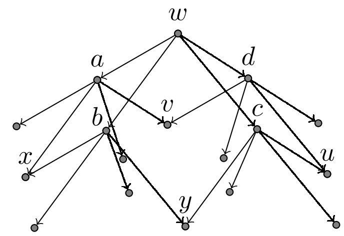
To prove a matching lower bound, we employ the apparatus of tiling systems. A *tiling system* is a triple $\langle C, H, V\rangle$, where C is a non-empty, finite set and H, V are binary relations on C. The elements of C are referred to as *colours*,



(a) The set of all z-worlds forming a (rather jumbled) ‘ziggurat’ under the direct successor relation. The world w_0, with character $(0,0)$, lies at the apex of the ziggurat, and the worlds with character (n,n) form its base.



(b) The direct successors of a z-world with character (i,j), where $0 \leq i < n$ and $0 \leq j < n$. Any such z-world has four direct successors: two with character $(i+1,j)$ and complementary values of p_{i+1}, and two with character $(i,j+1)$ and complementary values of q_{j+1}.



(c) Identifying z-worlds with the same indices using Formulas (8)–(10). From every z-world w with character (i,j), we can access at most two z-worlds a and c with character $(i+1,j)$, at most two z-worlds b and d with character $(i,j+1)$, and at most four (not eight!) z-worlds x, y, u and v with character $(i+1,j+1)$.

Figure 1. The set of z-worlds generated by Formulas (2)–(10).

and the relations H and V as the *horizontal* and *vertical* constraints, respectively. For any integer N, a *tiling* for $\langle C,H,V\rangle$ of size N is a function $f:\{0,\ldots,N-1\}^2 \to C$ such that, for all i,j with $0 \leq i < N-1$, $0 \leq j \leq N-1$, the pair $\langle f(i,j), f(i+1,j)\rangle$ is in H and for all i,j with $0 \leq i \leq N-1$, $0 \leq j < N-1$, the pair $\langle f(i,j), f(i,j+1)\rangle$ is in V. A tiling of size N is to be pictured as a colouring of an $N \times N$ square grid by the colours in C; the horizontal constraints H thus specify which colours may appear ‘to the right of’ which other colours; the vertical constraints V likewise specify which colours may appear ‘above’ which other colours. An n-tuple $\bar{c}$ of elements of C is an *initial configuration* for the tiling f if $\bar{c} = f(0,0),\ldots,f(n-1,0)$. An initial configuration for f is to be pictured as a row of n colours occupying the bottom left-hand corner of the grid.

Let (C,H,V) be a tiling system and p a polynomial. The *exponential tiling problem* (C,H,V,p) is the following problem: given an n-tuple $\bar{c}$ from C, determine whether there exists a tiling for (C,H,V) of size $2^{p(n)}$ with initial configuration $\bar{c}$. It is well-known that there exist exponential tiling problems which are NExpTime-complete (see, e.g. [19], pp. 242, ff.). We show how, for any class of frames $\mathcal{K}$ such that $\mathrm{Tr} \supseteq \mathcal{K} \supseteq \mathrm{Tr} \cap \mathrm{Rfl}$, any exponential tiling problem (C,H,V,p) can be reduced to $\mathcal{GM}_{\mathcal{K}}$-Sat, in polynomial time.

In the sequel, we denote by $\{0,1\}^*$ the set of finite strings over the alphabet $\{0,1\}$; we denote the length of any $s \in \{0,1\}^*$ by $\|s\|$; we denote the empty string by ϵ; and we write $s \preceq t$ if s is a (proper or improper) prefix of t. If $\|s\| = k$, then s encodes a number in the range $[0, 2^k - 1]$ in the usual way; we follow standard practice in taking the left-most digit of s to be the most significant. We equivocate freely between strings and the numbers they represent; in particular, we write $s+1$ to denote the string representing the successor of the number represented by s. Finally, if s is a string and $1 \leq k \leq \|s\|$, denote the kth element of s (counting from the left and starting with 1) by $s[k]$. We use the notation $\pm_i\varphi$ (with i a numerical subscript), to stand, ambiguously, for the formulas φ or $\neg\varphi$. All occurrences of $\pm_i\varphi$ within a single formula should be expanded in all possible ways to φ and $\neg\varphi$ such that occurrences with the same index i are expanded in the same way.

We are going to write formulas that induce a structure similar to that depicted in Fig. 1a, the bottom of which will represent the grid associated with (an instance of) a tiling problem. Fix $n > 0$. We consider structures interpreting the proposition letters $u_0,\ldots,u_n$, $v_0,\ldots,v_n$, $p_1,\ldots,p_n$, $q_1,\ldots,q_n$, z, o_h and o_v. Let Γ_1 be the set of all formulas:

$$u_0 \wedge v_0 \wedge z \tag{2}$$

$$\boxdot(\neg(u_i \wedge u_j) \wedge \neg(v_i \wedge v_j)) \qquad (0 \leq i < j \leq n) \tag{3}$$

$$\begin{aligned}&\boxdot(u_i \wedge v_j \wedge z \to && (0 \leq i < n,\\ &\quad \Diamond(u_{i+1} \wedge v_j \wedge z \wedge \pm_1 p_{i+1})) && \;0 \leq j \leq n)\end{aligned} \tag{4}$$

$$\begin{aligned}&\boxdot(u_i \wedge v_j \wedge z \to && (0 \leq i \leq n,\\ &\quad \Diamond(u_i \wedge v_{j+1} \wedge z \wedge \pm_1 q_{j+1})) && \;0 \leq j < n)\end{aligned} \tag{5}$$

$$\Box(u_i \wedge \pm_1 p_k \to \Box(z \to \pm_1 p_k)) \qquad (1 \leq k \leq i \leq n) \tag{6}$$

$$\Box(v_j \wedge \pm_1 q_k \to \Box(z \to \pm_1 q_k)) \qquad (1 \leq k \leq j \leq n) \tag{7}$$

Suppose $\mathfrak{A}$ is a transitive structure and w_0 a world of $\mathfrak{A}$ such that $\mathfrak{A} \models_{w_0} \Gamma_1$. We employ the following terminology. A world w of $\mathfrak{A}$ *has character* (i,j), for i,j in the range $[0,n]$, if $\mathfrak{A} \models_w u_i \wedge v_j$. A *z-world* is a member of the smallest set Z of worlds such that: (i) $w_0 \in Z$; and (ii) if $w \in Z$, and w' is a direct successor of w with $\mathfrak{A} \models_{w'} z$, then $w' \in Z$. (Notice that the definition of z-world depends on w_0; where w_0 is not clear from context, we speak of a z-world *relative to* w_0.) Necessarily, every z-world is either identical to, or accessible from, w_0. For any z-world w, with character (i,j), we define strings $s,t \in \{0,1\}^*$ of length i and j, respectively, by setting $s[k] = 1$ if and only if $\mathfrak{A} \models_w p_k$ for all k $(1 \leq k \leq i)$, and $t[k] = 1$ if and only if $\mathfrak{A} \models_w q_k$ for all k $(1 \leq k \leq j)$. The quadruple (i,j,s,t) is the *index* of w.

To see that Formulas (2)–(7) generate the structure in Fig. 1a, note first that Formula (2) implies the existence of a z-world w_0 with character $(0,0)$. Formulas (3) ensure that every z-world has a unique character. If $0 \le i < n$ and $0 \le j < n$, then Formulas (4) and (5) imply that every z-world with character (i,j) has four direct successors: two with character $(i+1,j)$ and complementary values of p_{i+1}, and two with character $(i,j+1)$ and complementary values of q_{j+1} (Fig. 1b). Similarly, if $0 \le i < n$ and $j = n$, or if $0 \le j < n$ and $i = n$, every z-world with character (i,j) has two direct successors.

Lemma 8. *Suppose $\mathfrak{A} \models_{w_0} \Gamma_1$. Let w be a z-world with index (i,j,s,t), and suppose i', j', s', t' satisfy: (i) $i \le i' \le n$; (ii) $j \le j' \le n$; (iii) $i+j < i'+j'$; (iv) $s \preceq s'$ and $\|s'\| = i'$; and (v) $t \preceq t'$ and $\|t'\| = j'$. Then there exists a z-world w', accessible from w, with index (i',j',s',t').*

Proof: Easy induction using Formulas (4)–(7). ■

Lemma 9. *Suppose $\mathfrak{A} \models_{w_0} \Gamma_1$. For all i ($0 \le i \le n$), all j ($0 \le j \le n$), all $s \in \{0,1\}^*$ ($\|s\| = i$) and all $t \in \{0,1\}^*$ ($\|t\| = j$), there exists a z-world with index (i,j,s,t).*

Proof: From Lemma 8 and the fact that w_0 has index $(0,0,\epsilon,\epsilon)$. ■

We now add formulas limiting the number of z-worlds with any given character (see Fig. 1c). In particular, z-worlds will turn out to be uniquely identified by their indices. Let Γ_2 be the set of formulas:

$$\boxdot(u_i \wedge v_j \to \Diamond_{\le 1}(u_{i+1} \wedge v_j \wedge \pm_1 p_{i+1})) \quad (0 \le i < n,\ 0 \le j \le n) \tag{8}$$

$$\boxdot(u_i \wedge v_j \to \Diamond_{\le 1}(u_i \wedge v_{j+1} \wedge \pm_1 q_{j+1})) \quad (0 \le i \le n,\ 0 \le j < n) \tag{9}$$

$$\boxdot(u_i \wedge v_j \to \Diamond_{\le 1}(u_{i+1} \wedge v_{j+1} \wedge \pm_1 p_{i+1} \wedge \pm_2 q_{j+1})) \quad (0 \le i < n,\ 0 \le j < n) \tag{10}$$

Lemma 10. *Suppose $\mathfrak{A} \models_{w_0} \Gamma_1 \cup \Gamma_2$. Then no two different z-worlds have the same index.*

Proof: Order the pairs of integers in the range $[0,n]$ in some way such that $i+j < i'+j'$ implies $(i,j) < (i',j')$, and proceed by induction on the character (i,j) of z-worlds, under this ordering.

Case 1: w has character $(0,0)$. By definition, w_0 is the only z-world with character $(0,0)$, and hence the only z-world with index $(0,0,\epsilon,\epsilon)$.

Case 2: w_1 and w_2 have index $(i+1,j+1,sa,tb)$ where, $0 \le i < n$, $0 \le j < n$ and $a,b \in \{0,1\}$. If w_1 and w_2 are z-worlds, there exist z-worlds w_1' and w_2' such that w_i is a direct successor of w_i' ($1 \le i \le 2$). The possible characters of w_1' and w_2' are $(i+1,j)$ and $(i,j+1)$. If w_1' and w_2' have the same character, then they in fact have the same index (this follows from Formulas (6) and (7), and the fact that w_1 and w_2 have the same index). By inductive hypothesis, then, $w_1' = w_2'$. Hence, from Formulas (8) or (9), $w_1 = w_2$ as required. If w_1' and w_2' have different characters, assume without loss of generality that w_1' has index $(i,j+1,s,tb)$, and w_2' has index $(i+1,j,sa,t)$. By Lemma 9, let w^* be any z-world with index (i,j,s,t). By Lemma 8, let w_1'' and w_2'' be z-worlds, accessible from w^*, with indices $(i,j+1,s,tb)$, and $(i+1,j,sa,t)$, respectively. By inductive hypothesis, $w_1' = w_1''$, and $w_2' = w_2''$: that is to say, w_1' and w_2' are accessible from w^*. Therefore, so are w_1 and w_2. Formulas (10) then ensure that $w_1 = w_2$.

Case 3: w_1 and w_2 have index $(i+1,0,sa,\epsilon)$ where $0 \le i < n$ and $a \in \{0,1\}$. The argument is similar to Case 2, and requires only Formulas (8).

Case 4: w_1 and w_2 have index $(0,j+1,\epsilon,tb)$ where $0 \le j < n$ and $b \in \{0,1\}$. The argument is similar to Case 2, and requires only Formulas (9). ■

Lemma 11. *Suppose $\mathfrak{A} \models_{w_0} \Gamma_1 \cup \Gamma_2$. Let w_1, w_2 be z-worlds with indices (i_1,j_1,s_1,t_1) and (i_2,j_2,s_2,t_2), respectively. Let s^* be a common prefix of s_1 and s_2, and t^* a common prefix of t_1 and t_2. Let $i^* = \|s^*\|$ and $j^* = \|t^*\|$. Then there exists a z-world w^* with index (i^*,j^*,s^*,t^*) such that each of w_1 and w_2 is either identical to, or accessible from, w^*.*

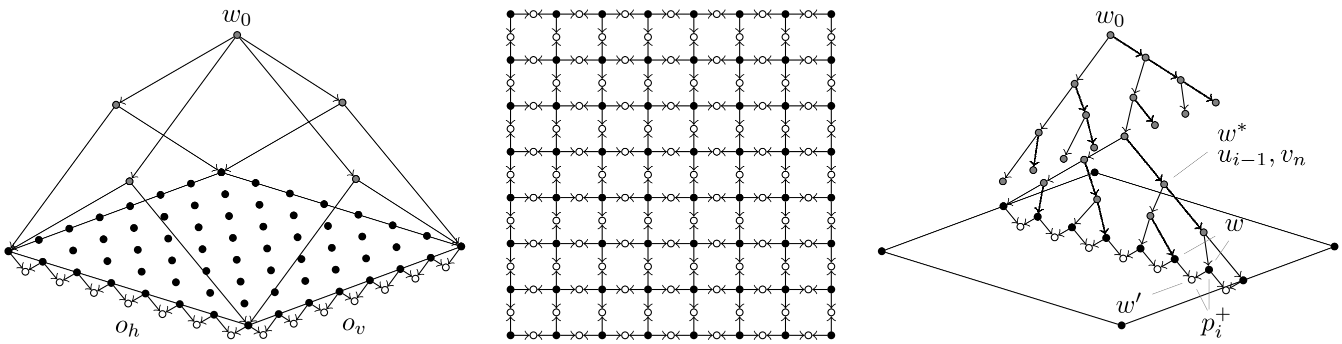
Proof: By Lemma 9 there exists a z-world w^* with index (i^*,j^*,s^*,t^*). If $i^*+j^* = i_1+j_1$ then $s^* = s_1$ and $t^* = t_1$, thus $w^* = w_1$ by Lemma 10. Otherwise $i^*+j^* < i_1+j_1$ and by Lemma 8, there exists a world w_1' accessible from w^* with index (i_1,j_1,s_1,t_1). By Lemma 10, $w_1' = w_1$. Thus w_1 is accessible from w^*. Similarly, one can show that either $w^* = w_2$ or w_2 is accessible from w^*. ■

The z-worlds of most interest are those with character (n,n)—of which, by Lemmas 9 and 10, there are exactly 2^{2n}. We refer to such worlds as *g-worlds* (g for 'grid').

For any world w (not just z-worlds), we define strings $s,t \in \{0,1\}^*$ of length n, by setting, for all k ($1 \le k \le n$), $s[k] = 1$ if and only if $\mathfrak{A} \models_w p_k$, and $t[k] = 1$ if and only if $\mathfrak{A} \models_w q_k$. We call the string s the *x-coordinate* of w, and the string t its *y-coordinate*. Notice that, if w is a g-world, with index (n,n,s,t), then its coordinates are (s,t). The strings s and t may of course be regarded as integers in the range $[0,2^n-1]$, and in the sequel we equivocate freely between strings of length n and the integers in this range they represent. The following abbreviations will be useful. If $1 \le i \le n$, we write p_i^* for $\neg p_i \wedge p_{i+1} \wedge \cdots \wedge p_n$, and p_i^+ for $p_i \wedge \neg p_{i+1} \wedge \cdots \wedge \neg p_n$. Thus, p_i^* and p_i^+ characterize those worlds whose x-coordinates are of the forms

$$a_1 \cdots a_{i-1} 0 \overbrace{1 \cdots\cdots 1}^{n-i \text{ times}} \qquad a_1 \cdots a_{i-1} 1 \overbrace{0 \cdots\cdots 0}^{n-i \text{ times}}, \tag{11}$$

respectively. Observe that, if s and s' are the respective strings (i.e. integers) depicted in (11), then $s' = s+1$. The abbreviations q_i^* and q_i^+ will be used similarly.



(a) The ziggurat, together with the grid at its base. (b) The world arrangement for the grid. (c) An illustration of Formulas (14) and Lemma 14.

Figure 2. Creating o-worlds (shown as a hollow dots) and the grid using Formulas (12)–(17) ($n = 3$): g-worlds (shown as filled dots) are arranged according to their coordinates at the base; g-worlds which are horizontal neighbours in this grid have a common horizontal o-world successor, while g-worlds which are vertical neighbours in this grid have a common vertical o-world successor.

We now write formulas which force the g-worlds to link up into a $2^n \times 2^n$ grid (see Fig. 2). This process is complicated by the fact that we are dealing with transitive accessibility relations. We employ proposition letters o_h, o_v, and refer to worlds satisfying these proposition letters as, respectively, *horizontal o-worlds* and *vertical o-worlds* ('o' stands for nothing in particular). The o-worlds' function is to glue the g-worlds into the desired grid pattern. Let $\Gamma_{3,h}$ be the set of formulas:

$$\Box(u_n \wedge v_n \wedge p_i^* \rightarrow \Diamond(o_h \wedge p_i^+)) \qquad (1 \leq i \leq n) \qquad (12)$$
$$\Box(u_n \wedge v_n \wedge p_i^+ \rightarrow \Diamond(o_h \wedge p_i^+)) \qquad (1 \leq i \leq n) \qquad (13)$$
$$\Box(u_{i-1} \wedge v_n \rightarrow \Diamond_{\leq 1}(o_h \wedge p_i^+)) \qquad (1 \leq i \leq n), \qquad (14)$$

and suppose $\mathfrak{A} \models_{w_0} \Gamma_1 \cup \Gamma_2 \cup \Gamma_{3,h}$. Consider a g-world w with coordinates (s,t). If $0 \leq s < 2^{n-1}$, then w satisfies p_i^* for some $i > 0$, and so has a horizontal o-world successor by Formulas (12); likewise, if $0 < s \leq 2^n - 1$, then w satisfies p_i^+ for some $i > 0$, and so has a horizontal o-world successor by Formulas (13). (Hence, if $0 < s < 2^{n-1}$, then w has at least two horizontal o-world successors.) Finally, let i be such that $1 \leq i \leq n$, and suppose that w^* is a z-world with character $(i-1, n)$. Formulas (14) imply that there is at most one horizontal o-world accessible from w^*, and satisfying p_i^+ (see Fig. 2c). The effect of these sets of formulas is illustrated in Fig. 2 and formalized in the following lemma:

Lemma 12. *Suppose $\mathfrak{A} \models_{w_0} \Gamma_1 \cup \Gamma_2 \cup \Gamma_{3,h}$. Let w and w' be g-worlds with coordinates (s,t) and $(s+1,t)$, respectively. Then there exists a horizontal o-world u accessible from both w and w' such that $\mathfrak{A} \models_u p_n$ if and only if $\mathfrak{A} \models_{w'} p_n$.*

Proof: Since $0 \leq s < s+1 \leq 2^n - 1$, there exists i such that w satisfies p_i^*; thus w' satisfies p_i^+. From Formulas (12) and (13), there exist o-worlds u, u' both satisfying p_i^+, with u accessible from w, and u' accessible from w'. Clearly, $\mathfrak{A} \models_u p_n$ if and only if $\mathfrak{A} \models_{w'} p_n$. By Lemma 11, there exists a z-world w^* with character $(i-1, n)$, for some i ($1 \leq i \leq n$), such that both w and w', and hence both u and u', are accessible from w^*. From Formulas (14), we have $u = u'$. ■

Similarly, let $\Gamma_{3,v}$ be the set of formulas:

$$\Box(u_n \wedge v_n \wedge q_i^* \rightarrow \Diamond(o_v \wedge q_i^+)) \qquad (1 \leq i \leq n) \qquad (15)$$
$$\Box(u_n \wedge v_n \wedge q_i^+ \rightarrow \Diamond(o_v \wedge q_i^+)) \qquad (1 \leq i \leq n) \qquad (16)$$
$$\Box(u_n \wedge v_{i-1} \rightarrow \Diamond_{\leq 1}(o_v \wedge q_i^+)) \qquad (1 \leq i \leq n). \qquad (17)$$

Lemma 13. *Suppose $\mathfrak{A} \models_{w_0} \Gamma_1 \cup \Gamma_2 \cup \Gamma_{3,v}$. Let w and w' be g-worlds with coordinates (s,t) and $(s,t+1)$, respectively. Then there exists a vertical o-world u accessible from both w and w' such that $\mathfrak{A} \models_u q_n$ if and only if $\mathfrak{A} \models_{w'} q_n$.*

Proof: Analogous to Lemma 12. ■

Let $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_{3,h} \cup \Gamma_{3,v}$, and suppose $\mathfrak{A} \models_{w_0} \Gamma$. Lemmas 9 and 10 guarantee that, for all s, t in the range $[0, 2^n - 1]$, there exists exactly one g-world with coordinates (s,t); let G be the set of all these 2^{2n} g-worlds. And let O_v, O_h be sets of horizontal and vertical o-worlds guaranteed by Lemmas 12 and 13, respectively. Thus, the frame of $\mathfrak{A}$ contains, as a subgraph, the configuration depicted in Fig. 2b. In short, the formulas Γ manufacture a $2^n \times 2^n$ grid.

Conversely, it is easy to exhibit a model of Γ, using the diagrams of Fig. 2 as our guide, containing just such a grid.

Lemma 14. *There exists a structure $\mathfrak{S}$ over a reflexive, transitive frame, and a world w_0 of $\mathfrak{S}$, such that $\mathfrak{S} \models_{w_0} \Gamma$.*

Proof: For h and v distinct symbols, define the sets:

$$\begin{aligned}
Z &= \{(i,j,s,t) \mid 0 \leq i \leq n;\ 0 \leq j \leq n; \\
&\qquad s,t \in \{0,1\}^*; \|s\| = i \text{ and } \|t\| = j\} \\
G &= \{(n,n,s,t) \mid s,t \in \{0,1\}^* \text{ and } \|s\| = \|t\| = n\} \\
O_h &= \{(h,s,t) \mid s,t \in \{0,1\}^*; s \notin \{0\}^*; \|s\| = \|t\| = n\} \\
O_v &= \{(v,s,t) \mid s,t \in \{0,1\}^*; t \notin \{0\}^*; \|s\| = \|t\| = n\}.
\end{aligned}$$

Note that $G \subseteq Z$. Define the binary relations $R_Z \subseteq Z \times Z$, $R_h \subseteq G \times O_h$ and $R_v \subseteq G \times O_v$ by:

$$\begin{aligned}
R_Z &= \{\langle (i,j,s,t),(i',j',s',t')\rangle \\
&\qquad \mid i \leq i';\ j \leq j';\ s \preceq s \text{ and } t \preceq t'\} \\
R_h &= \{\langle (n,n,s,t),(h,s',t')\rangle \\
&\qquad \mid t' = t;\ s \leq s' \leq n \text{ and } 1 \leq s' \leq s+1\} \\
R_v &= \{\langle (n,n,s,t),(v,s',t')\rangle \\
&\qquad \mid s' = s;\ t \leq t' \leq n \text{ and } 1 \leq t' \leq t+1\}.
\end{aligned}$$

Finally, let $S = Z \cup O_h \cup O_v$, and let R_S be the reflexive, transitive closure of $R_Z \cup R_h \cup R_v$. Thus, (S, R_S) is a reflexive, transitive frame. Define a valuation V on (S, R_S) by interpreting the proposition letters as follows:

$$\begin{aligned}
z^{\mathfrak{S}} &= Z;\quad o_h^{\mathfrak{S}} = O_h;\quad o_v^{\mathfrak{S}} = O_v \\
u_i^{\mathfrak{S}} &= \{(i,j,s,t) \in Z \mid 0 \leq j \leq n;\ s,t \in \{0,1\}^*\} \\
v_j^{\mathfrak{S}} &= \{(i,j,s,t) \in Z \mid 0 \leq i \leq n;\ s,t \in \{0,1\}^*\} \\
p_i^{\mathfrak{S}} &= \{(i',j,s,t) \in Z \mid i' \geq i,\ s[i] = 1\} \cup \\
&\quad \{(h,s,t) \in O_h \mid s[i] = 1\} \cup \\
&\quad \{(v,s,t) \in O_v \mid s[i] = 1\} \\
q_j^{\mathfrak{S}} &= \{(i,j',s,t) \in Z \mid j' \geq j,\ t[j] = 1\} \cup \\
&\quad \{(h,s,t) \in O_h \mid t[j] = 1\} \cup \\
&\quad \{(v,s,t) \in O_v \mid t[j] = 1\}.
\end{aligned}$$

Denote by $\mathfrak{S}$ the structure (S, R_S, V). Let $w_0 \in Z$ be the element $(0,0,\epsilon,\epsilon)$. Thus, $\mathfrak{S} \models_{w_0} \Gamma_1$, and, relative to w_0, the z-worlds of $\mathfrak{S}$ are simply the elements of Z. It is obvious that, for every $w = (i,j,s,t) \in Z$, the index of w is w itself; moreover, for every $w = (h,s,t) \in o_h$ and every $w = (v,s,t) \in o_v$, the coordinates of w are (s,t).

We now show that $\mathfrak{S} \models_{w_0} \Gamma$. The truth at w_0 of Formulas (2)–(17) except for Formulas (14) and (17) is immediate. To demonstrate the truth of Formulas (14), let $1 \leq i \leq n$, and fix any world w^* of $\mathfrak{S}$ such that $\mathfrak{S} \models_{w^*} u_{i-1} \wedge v_n$ (see Fig. 2c). We may write $w^* = (i-1, n, s^*, t^*)$, where $\|s^*\| = i-1$ and $\|t^*\| = n$. Now suppose w' is any world of $\mathfrak{S}$ such that $\langle w^*, w'\rangle \in R_S$ and $\mathfrak{S} \models_{w'} o_h \wedge p_i^+$. Again, we may write $w' = (h, s', t')$, where s' and t' are bit-strings of length n. We claim that $s' = s^*10\ldots0$ and $t' = t^*$. But there is at most one world in $\mathfrak{S}$ satisfying o_h and having coordinates $(s^*10\ldots0, t^*)$; hence, $\mathfrak{S} \models_{w_0} \Box(u_{i-1} \wedge v_n \rightarrow \Diamond_{\leq 1}(o_h \wedge p_i^+))$, as required.

To prove the claim, observe that, by construction of $\mathfrak{S}$, there exists $w \in G$ such that $\langle w^*, w\rangle \in R_S$ and $\langle w, w'\rangle \in R_S$. Pick any such w and let it have coordinates (s,t). By the definition of R_S (and the fact that $\|t^*\| = n$), we have: (i) $t^* = t = t'$, (ii) $s^* \preceq s$, and (iii) $s' = s$ or $s' = s+1$. Referring to Fig. 2c, the worlds w^*, w and w' can be reached from w_0 by traversing two trees of z-worlds: an upper tree, whose leaves have characters $(0,n)$, and a lower tree, whose elements have characters (i,n) $(0 \leq i \leq n)$. The world w^* in the lower tree, has character $(i-1,n)$; w' is a horizontal o-world reachable from w^*; w is its predecessor g-world. Now, since $\mathfrak{S} \models_{w'} o_h \wedge p_i^+$, we have $s' = s''10\ldots0$ for some string s'' with $\|s''\| = i-1$. Since s is either s' or $s'-1$, we have either $s = s''10\ldots0$ or $s = s''01\ldots1$. Since $s^* \preceq s$ and $\|s^*\| = i-1$, we have $s'' = s^*$. Thus, $s' = s^*10\ldots0$ and $t' = t^*$, proving the claim.

The case of Formulas (17) is treated analogously. ∎

Now we are in a position to encode any exponential tiling problem, (C,H,V,p) in our logic. We regard colours $c \in C$ as (fresh) proposition letters. Suppose $\mathfrak{A}$ is transitive and $\mathfrak{A} \models_{w_0} \Gamma$, and let $\mathfrak{A}$ additionally interpret the proposition letters $c \in C$. By Lemmas 9, 10, 12, and 13, the frame of $\mathfrak{A}$ contains the arrangement of Fig. 2b as a subgraph, which we may partition into the sets G (the g-worlds), O_h (the horizontal o-worlds) and O_v (the vertical o-worlds). Intuitively, for any world $w \in G$, c represents the colour of w in some (putative) tiling of G. Now we write formulas to ensure that the colours form a tiling for (C,H,V,p). Define Δ to be the following set of formulas:

$$\Box\left(u_n \wedge v_n \rightarrow \left(\bigvee C \wedge \bigwedge\{\neg c \vee \neg d \mid c \neq d\}\right)\right) \tag{18}$$

$$\begin{aligned}
&\Box(u_n \wedge v_n \wedge \pm_1 p_n \wedge c \rightarrow \\
&\quad \Box(o_h \wedge \pm_1 p_n \rightarrow c)) \qquad (c \in C)
\end{aligned} \tag{19}$$

$$\begin{aligned}
&\Box(u_n \wedge v_n \wedge \pm_1 p_n \wedge c \rightarrow \\
&\quad \Box(o_h \wedge \neg(\pm_1 p_n) \rightarrow \neg d)) \qquad (\langle c,d\rangle \notin H)
\end{aligned} \tag{20}$$

$$\begin{aligned}
&\Box(u_n \wedge v_n \wedge \pm_1 q_n \wedge c \rightarrow \\
&\quad \Box(o_v \wedge \pm_1 q_n \rightarrow c)) \qquad (c \in C)
\end{aligned} \tag{21}$$

$$\begin{aligned}
&\Box(u_n \wedge v_n \wedge \pm_1 q_n \wedge c \rightarrow \\
&\quad \Box(o_v \wedge \neg(\pm_1 q_n) \rightarrow \neg d)) \qquad (\langle c,d\rangle \notin V).
\end{aligned} \tag{22}$$

Formula (18) ensures that every g-world is assigned a unique colour. Using Lemma 12, Formulas (19) ensure every horizontal o-world has the same colour as the g-world 'immediately to the right'. Together with Formulas (18) and (20), this ensures that the g-worlds satisfy the horizontal tiling constraints. Likewise, Formulas (18), (21), and (22) ensure that the g-worlds satisfy the vertical tiling constraints.

Lemma 15. *Suppose $\mathfrak{A}$ is transitive, and $\mathfrak{A} \models_{w_0} \Gamma \cup \Delta$. For all s,t in the range $[0, 2^n - 1]$, define $f(s,t) = c$ if $\mathfrak{A} \models_w c$ for some g-world w with coordinates (s,t). Then f is well-defined, and is in fact a tiling for (C,H,V).*

Proof: Immediate. ∎

Now suppose $\bar{d} = d_0, \ldots, d_{m-1}$ is an m-tuple of elements of C. Let π_0 be the formula:

$$\Box(z \wedge \neg p_1 \wedge \cdots \wedge \neg p_n \wedge \neg q_1 \wedge \cdots \wedge \neg q_n \rightarrow d_0)$$

implying that any g-world with coordinates $(0,0)$ has colour d_0; and let the formulas $\pi_1, \ldots, \pi_{m-1}$ be defined analogously, assigning colours $d_1, \ldots, d_{m-1}$ to the g-worlds with coordinates $(1,0), \ldots, (m-1,0)$. Denote by $\Theta_{\bar{d}}$ the set of all these formulas.

Lemma 16. *Suppose $\mathfrak{A}$ is transitive, with $\mathfrak{A} \models_{w_0} \Gamma \cup \Delta \cup \Theta_{\bar{d}}$, and let the tiling f be as defined in Lemma 15. Then $\bar{d}$ is an initial configuration for f.*

Proof: Immediate. ■

Thus, we have:

Lemma 17. *Let $\mathcal{K}$ be any class of frames satisfying* Tr $\supseteq \mathcal{K} \supseteq$ Tr $\cap$ Rfl. *The problem* $\mathcal{GM}_{\mathcal{K}}$-Sat *is* NExpTime-*hard. It remains* NExpTime-*hard, even when all numerical subscripts in modal operators are bounded by* 1.

Proof: We reduce any exponential tiling problem (C, H, V, p) to the problem $\mathcal{GM}_{\mathcal{K}}$-Sat. Fix (C, H, V, p), and let an instance $\bar{d}$ of size m be given. Write $n = p(m)$. Consider the conjunction $\varphi_{\bar{d}}$ of all formulas in the set $\Gamma \cup \Delta \cup \Theta_{\bar{d}}$. We claim that the following are equivalent: (*i*) $\varphi_{\bar{d}}$ is satisfiable over Tr $\cap$ Rfl; (*ii*) $\varphi_{\bar{d}}$ is satisfiable over Tr; (*iii*) $\bar{d}$ is a positive instance of (C, H, V, p). The implication (*i*) $\Rightarrow$ (*ii*) is trivial. For (*ii*) $\Rightarrow$ (*iii*), suppose $\mathfrak{A} \models_{w_0} \Gamma \cup \Delta \cup \Theta_{\bar{d}}$, with $\mathfrak{A}$ transitive. Lemmas 15 and 16 then guarantee the existence of a tiling f of size 2^n for (C, H, V), with initial configuration $\bar{d}$. For (*iii*) $\Rightarrow$ (*i*), suppose f is a tiling for (C, H, V) of size 2^n, with initial configuration $\bar{d}$. Taking $\mathfrak{S}$ and w_0 to be as in the proof of Lemma 14, we expand $\mathfrak{S}$ to a structure $\mathfrak{S}^*$ by setting $c^{\mathfrak{S}^*} = \{(n, n, s, t), (h, s, t), (v, s, t) \mid f(s, t) = c\}$ for every proposition letter $c \in C$. It is obvious that $\mathfrak{S}^* \models_{w_0} \Delta \cup \Theta_{\bar{d}}$. ■

Theorem 5 follows from Corollary 1 and Lemma 17, noting that Rfl $\cap$ Tr $=$ Rfl $\cap$ Ser $\cap$ Tr $\subseteq$ Ser $\cap$ Tr $\subseteq$ Tr.

V. Conclusion

In this paper, we have investigated the computational complexity of $\mathcal{GM}_{\cap\mathcal{F}}$-Sat, the satisfiability problem for graded modal logic over any frame class $\bigcap \mathcal{F}$, where $\mathcal{F} \subseteq \{\text{Rfl}, \text{Ser}, \text{Sym}, \text{Tr}, \text{Eucl}\}$. The results are as follows. Suppose first that Eucl $\notin \mathcal{F}$ and Tr $\notin \mathcal{F}$. Then Theorem 3 states that $\mathcal{GM}_{\cap\mathcal{F}}$-Sat is PSpace-complete. Suppose next that Eucl $\in \mathcal{F}$ or $\{\text{Sym}, \text{Tr}\} \subseteq \mathcal{F}$. Then Theorem 4 states that $\mathcal{GM}_{\cap\mathcal{F}}$-Sat is NP-complete. Suppose finally that Eucl, Sym $\notin \mathcal{F}$, but Tr $\in \mathcal{F}$. Then Theorem 5 states that $\mathcal{GM}_{\cap\mathcal{F}}$-Sat is NExpTime-complete. All these results hold under both unary and binary coding of numerical subscripts.

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