Approximately Bisimilar Symbolic Models for Stochastic Switched Systems

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Abstract. Stochastic switched systems are a class of continuous-time dynamical models with probabilistic evolution over a continuous domain and control-dependent discrete dynamics over a finite set of locations (modes). As such, they represent a subclass of general stochastic hybrid systems. While the literature has witnessed recent progress in the dynamical analysis and controller synthesis for the stability of stochastic switched systems, more complex and challenging objectives related to the verification of and the synthesis for logic specifications (properties expressed as formulas in linear temporal logic or as automata on infinite strings) have not been formally investigated as of yet. This paper addresses these complex objectives by constructively deriving approximately equivalent (bisimilar) symbolic models of stochastic switched systems. More precisely, a finite symbolic model that is approximately bisimilar to a stochastic switched system is constructed under some dynamical stability assumptions on the concrete model. This allows formally synthesizing controllers (switching signals) valid for the concrete system over the finite symbolic model, by means of mature techniques in the literature.

1 Introduction

Stochastic hybrid systems are general dynamical systems comprising continuous and discrete dynamics interleaved with probabilistic noise and stochastic events [7]. Because of their versatility and generality they carry great promise in many safety critical applications [7], including power networks, automotive and financial engineering, air traffic control, biology, telecommunications, and embedded systems. Stochastic *switched* systems are a relevant class of stochastic hybrid systems: they consist of a finite set of modes of operation (locations), each of which is associated to a probabilistic dynamical behavior; further, their discrete dynamics, in the form of location changes, are governed by a deterministic control signal. However unlike general stochastic hybrid systems they do not present probabilistic discrete dynamics (random switch of locations), nor continuous resets upon location change.

It is known [18, 21] that switched systems can be endowed with global dynamics that are not characteristic of the behavior of any of their modes: for instance, global instability can arise by proper choice of the discrete switches between a set of stable dynamical locations. This is one of the many features that makes switched systems theoretically interesting. With focus on *stochastic* switched systems, despite recent progress on basic dynamical analysis focused on stability properties [9], there are no notable results in terms of more complex objectives, such as those dealing with verification or (controller) synthesis for logical specifications. Specifications of interest are expressed as formulas in linear temporal logic or via automata on infinite strings, and as such they are not amenable to classical approaches from the literature on stochastic processes.

A promising direction to investigate these general properties is the use of *symbolic models*. Symbolic models are abstract descriptions of the original dynamics, where each abstract state (or symbol) corresponds to an aggregate of states in the concrete system. When a finite symbolic model is obtained and formally put in relationship with the original system, one can leverage mature techniques for controller synthesis over the discrete model [12, 20, 30] to automatically synthesize controllers for the original system. Towards this goal, a relevant approach is the construction of finite-state symbolic models that are *bisimilar* to the original system. Unfortunately, the class of continuous (time and space) dynamical systems admitting exactly bisimilar finite-state symbolic models is quite restrictive [4, 17, 25] and in particular it covers exclusively non-probabilistic models. Therefore, rather than requiring systems equivalence, one can resort to *approximate bisimulation* relations [14], which introduce a metric between the trajectories of the abstract and the concrete models, and require boundedness in time of this distance.

The construction of approximately bisimilar symbolic models has been recently studied for non-probabilistic continuous control systems, possibly endowed with non-determinism [19, 26, 27], as well as for non-probabilistic switched systems [15], but stochastic systems, particularly when endowed with switched dynamics, have only been partially explored. With focus on stochastic systems, few existing results deal with abstractions of discrete-time processes [2, 3, 6], whereas results for continuous-time models are limited to probabilistic rectangular hybrid automata [28] and stochastic dynamical systems under some contractivity assumptions [1]. In summary to the best of our knowledge, there is no comprehensive work on the construction of finite bisimilar abstractions for continuous-time stochastic systems with control actions or with switched dynamics. A recent result [31] by the authors investigates this goal over stochastic systems with no discrete dynamics.

The main contribution of this work consists in showing the existence and the construction of approximate bisimilar symbolic models for incrementally stable stochastic switched systems. Incremental stability is a stability assumption applied to the stochastic switched systems under study: it can be characterized in terms of a so-called Lyapunov function (which we shall see can either be a single global function or a set of location-dependent ones). It is an extension of a similar notion developed for non-probabilistic switched systems [15] in the sense that the results for non-probabilistic switched systems represent a special case of the results in this paper when the continuous dynamics are degenerate (they present no noise). The results are illustrated over the synthesis of controllers (location switching signals) for two examples. First, we consider a room temperature control problem (admitting a global – or common – Lyapunov function) that is subject to a constraint expressed by a finite automaton, and show a switched controller synthesis for the temperature regulation toward a desired level. The second example illustrates the use of multiple Lyapunov functions (one per location). The proof of the statements in this work are provided in the Appendix.

2 Stochastic Switched Systems

2.1 Notation

The identity map on a set A is denoted by 1_A. If A is a subset of B, we denote by $\imath_A : A \hookrightarrow B$ or simply by $\imath$ the natural inclusion map taking any $a \in A$ to $\imath(a) = a \in B$. The symbols $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ denote the set of natural, nonnegative integer, integer, real, positive, and nonnegative real numbers, respectively. The symbols I_n, 0_n, and $0_{n\times m}$ denote the identity matrix, zero vector, and zero matrix in $\mathbb{R}^{n\times n}$, $\mathbb{R}^n$, and $\mathbb{R}^{n\times m}$, respectively. Given a vector $x \in \mathbb{R}^n$, we denote by x_i the i–th element of x, and by $\|x\|$ the infinity norm of x, namely, $\|x\| = \max\{|x_1|, |x_2|, ..., |x_n|\}$, where $|x_i|$ denotes the absolute value of x_i. Given a matrix $M = \{m_{ij}\} \in \mathbb{R}^{n\times m}$, we denote by $\|M\|$ the infinity norm of M, namely, $\|M\| = \max_{1\leq i\leq n} \sum_{j=1}^{m} |m_{ij}|$, and by $\|M\|_F$ the Frobenius norm of M, namely, $\|M\|_F = \sqrt{\text{Tr}\,(MM^T)}$, where $\text{Tr}(P) = \sum_{i=1}^{n} p_{ii}$ for any $P = \{p_{ij}\} \in \mathbb{R}^{n\times n}$. Notations $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ stand for the minimum and maximum eigenvalues of matirx A, respectively.

The closed ball centered at $x \in \mathbb{R}^n$ with radius ε is defined by $\mathcal{B}_\varepsilon(x) = \{y \in \mathbb{R}^n \mid \|x - y\| \leq \varepsilon\}$. A set $B \subseteq \mathbb{R}^n$ is called a *box* if $B = \prod_{i=1}^{n}[c_i, d_i]$, where $c_i, d_i \in \mathbb{R}$ with $c_i < d_i$ for each $i \in \{1, \ldots, n\}$. The *span* of a box B is defined as $span(B) = \min\{|d_i - c_i| \mid i = 1, \ldots, n\}$. For a box B and $\eta \leq span(B)$, define the η-approximation $[B]_\eta = \{b \in B \mid b_i = k_i\eta$ for some $k_i \in \mathbb{Z}, i = 1, \ldots, n\}$. Note that $[B]_\eta \neq \varnothing$ for any $\eta \leq span(B)$. Geometrically, for any $\eta \in \mathbb{R}^+$ with $\eta \leq span(B)$ and $\lambda \geq \eta$ the collection of sets $\{\mathcal{B}_\lambda(p)\}_{p\in[B]_\eta}$ is a finite covering of B, i.e., $B \subseteq \bigcup_{p\in[B]_\eta} \mathcal{B}_\lambda(p)$. By defining $[\mathbb{R}^n]_\eta = \{a \in \mathbb{R}^n \mid a_i = k_i\eta, k_i \in \mathbb{Z}, i = 1, \cdots, n\}$, the set $\bigcup_{p\in[\mathbb{R}^n]_\eta} \mathcal{B}_\lambda(p)$ is a countable covering of $\mathbb{R}^n$ for any $\eta \in \mathbb{R}^+$ and $\lambda \geq \eta$. We extend the notions of $span$ and approximation to finite unions of boxes as follows. Let $A = \bigcup_{j=1}^{M} A_j$, where each A_j is a box. Define $span(A) = \min\{span(A_j) \mid j = 1, \ldots, M\}$, and for any $\eta \leq span(A)$, define $[A]_\eta = \bigcup_{j=1}^{M}[A_j]_\eta$.

Given a set X, a function $\mathbf{d} : X \times X \to \mathbb{R}_0^+$ is a metric on X if for any $x, y, z \in X$, the following three conditions are satisfied: i) $\mathbf{d}(x, y) = 0$ if and only if $x = y$; ii) $\mathbf{d}(x, y) = \mathbf{d}(y, x)$; and iii) (triangle inequality) $\mathbf{d}(x, z) \leq \mathbf{d}(x, y) + \mathbf{d}(y, z)$. A continuous function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$; γ is said to belong to class $\mathcal{K}_\infty$ if $\gamma \in \mathcal{K}$ and $\gamma(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the map $\beta(r, s)$ belongs to class $\mathcal{K}_\infty$ with respect to r and, for each fixed nonzero r, the map $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$. We identify a relation $R \subseteq A \times B$ with the map $R : A \to 2^B$ defined by $b \in R(a)$ iff $(a, b) \in R$. Given a relation $R \subseteq A \times B$, R^{-1} denotes the inverse relation defined by $R^{-1} = \{(b, a) \in B \times A : (a, b) \in R\}$.

2.2 Stochastic switched systems model

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with a filtration $\mathbb{F} = (\mathcal{F}_s)_{s\geq 0}$ satisfying the usual conditions of completeness and right-continuity [16, p. 48]. Let $(W_s)_{s\geq 0}$ be a q-dimensional $\mathbb{F}$-Brownian motion [24].

Definition 1. *A stochastic switched system is a tuple $\Sigma = (\mathbb{R}^n, \mathsf{P}, \mathcal{P}, F, G)$, where*

- $\mathbb{R}^n$ *is the continuous state space;*
- $\mathsf{P} = \{1, \cdots, m\}$ *is a finite set of modes, or locations;*
- $\mathcal{P}$ *is a subset of $\mathcal{S}(\mathbb{R}_0^+, \mathsf{P})$, which denotes the set of piecewise constant functions (by convention continuous from the right) from $\mathbb{R}_0^+$ to P, and characterized by a finite number of discontinuities on every bounded interval in $\mathbb{R}_0^+$;*
- *$F = \{f_1, \cdots, f_m\}$ such that, for all $p \in \mathsf{P}$, $f_p : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function satisfying the following Lipschitz assumption: there exists a constant $L \in \mathbb{R}^+$ such that, for all $x, x' \in \mathbb{R}^n$: $\|f_p(x) - f_p(x')\| \leq L\|x - x'\|$;*
- *$G = \{g_1, \cdots, g_m\}$ such that for all $p \in \mathsf{P}$, $g_p : \mathbb{R}^n \to \mathbb{R}^{n\times q}$ is a continuous function satisfying the following Lipschitz assumption: there exists a constant $Z \in \mathbb{R}^+$ such that for all $x, x' \in \mathbb{R}^n$: $\|g_p(x) - g_p(x')\| \leq Z\|x - x'\|$.*

Let us discuss the semantics of model Σ. For any given $p \in \mathsf{P}$, we denote by Σ_p the subsystem of Σ defined by the stochastic differential equation

$$\mathrm{d}\,\xi = f_p(\xi)\,\mathrm{d}\,t + g_p(\xi)\,\mathrm{d}\,W_t, \tag{1}$$

where f_p is known as the drift, g_p as the diffusion, and again W_t is Brownian motion. A solution process of Σ_p exists and is uniquely determined owing to the assumptions on f_p and on g_p [24, Theorem 5.2.1, p. 68].

For the global model Σ, a continuous-time stochastic process $\xi : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}^n$ is said to be a *solution process* of Σ if there exists a switching signal $\upsilon \in \mathcal{P}$ satisfying

$$\mathrm{d}\,\xi = f_\upsilon(\xi)\,\mathrm{d}\,t + g_\upsilon(\xi)\,\mathrm{d}\,W_t, \tag{2}$$

$\mathbb{P}$-almost surely ($\mathbb{P}$-a.s.) at each time $t \in \mathbb{R}_0^+$ when υ is constant. Let us emphasize that υ is a piecewise constant function defined over $\mathbb{R}_0^+$ and taking values in P, which simply dictates in which location the solution process ξ is at any time $t \in \mathbb{R}_0^+$. Notice that the mode changes are deterministic in that they are fully encompassed by a given function υ in $\mathcal{P}$ and that, whenever a location is changed (discontinuity in υ), the value of the process ξ is not reset on $\mathbb{R}^n$ – thus ξ is a continuous function of time.

We further write $\xi_{a\upsilon}(t)$ to denote the value of the solution process at time $t \in \mathbb{R}_0^+$ under the switching signal υ from initial condition $\xi_{a\upsilon}(0) = a$ $\mathbb{P}$-a.s., in which a is a random variable that is measurable in $\mathcal{F}_0$. Note that in general the stochastic switched system Σ may start from a random initial condition.

Finally, note that a solution process of Σ_p is also a solution process of Σ corresponding to the constant switching signal $\upsilon(t) = p$, for all $t \in \mathbb{R}_0^+$. We also use $\xi_{ap}(t)$ to denote the value of the solution process of Σ_p at time $t \in \mathbb{R}_0^+$ from the initial condition $\xi_{ap}(0) = a$ $\mathbb{P}$-a.s.

3 Notions of Incremental Stability

This section introduces some stability notions for stochastic switched systems, which generalize the concepts of incremental global asymptotic stability (δ-GAS) [5] for dynamical systems and of incremental global uniform asymptotic stability (δ-GUAS) [15] for non-probabilistic switched systems. The main results presented in this work rely on the stability assumptions discussed in this section.

Definition 2. *The stochastic subsystem Σ_p is incrementally globally asymptotically stable in the qth mean (δ-GAS-M$_q$), where $q \in \mathbb{N}$, if there exists a $\mathcal{KL}$ function β_p such that for any $t \in \mathbb{R}_0^+$, and any $\mathbb{R}^n$-valued random variables a and a' that are measurable in $\mathcal{F}_0$, the following condition is satisfied:*

$$\mathbb{E}\left[\|\xi_{ap}(t) - \xi_{a'p}(t)\|^q\right] \leq \beta_p\left(\mathbb{E}\left[\|a - a'\|^q\right], t\right). \tag{3}$$

Intuitively, the notion requires (a higher moment of) the distance between trajectories to be bounded and decreasing in time. It can be easily checked that a δ-GAS-M$_q$ stochastic subsystem Σ_p is δ-GAS [5] in the absence of any noise. Further, note that when $f_p(0_n) = 0_n$ and $g_p(0_n) = 0_{n\times q}$ (drift and diffusion terms vanish at the origin), then δ-GAS-M$_q$ implies GAS-M$_q$ [9], which means that all the trajectories of Σ_p converge in the qth mean to the (constant) trajectory $\xi_{0_n p}(t) = 0_n$ (the origin). We extend the notion of δ-GAS-M$_q$ to stochastic switched systems as follows.

Definition 3. *A stochastic switched system $\Sigma = (\mathbb{R}^n, \mathsf{P}, \mathcal{P}, F, G)$ is incrementally globally uniformly asymptotically stable in the qth mean (δ-GUAS-M$_q$), where $q \in \mathbb{N}$, if there exists a $\mathcal{KL}$ function β such that for any $t \in \mathbb{R}_0^+$, any $\mathbb{R}^n$-valued random variables a and a' that are measurable in $\mathcal{F}_0$, and any switching signal $\upsilon \in \mathcal{P}$, the following condition is satisfied:*

$$\mathbb{E}\left[\|\xi_{a\upsilon}(t) - \xi_{a'\upsilon}(t)\|^q\right] \leq \beta\left(\mathbb{E}\left[\|a - a'\|^q\right], t\right). \tag{4}$$

Essentially Definition 3 extends Definition 2 uniformly over any possible switching signal υ. As expected, the notion generalizes known ones in the literature: it can be easily seen that a δ-GUAS-M$_q$ stochastic switched system Σ is δ-GUAS [15] in the absence of any noise and that, whenever $f_p(0_n) = 0_n$ and $g_p(0_n) = 0_{n\times q}$ for all $p \in \mathsf{P}$, then δ-GUAS-M$_q$ implies GUAS-M$_q$ [9].

For non-probabilistic systems the δ-GAS property can be characterized by scalar functions defined over the state space, known as Lyapunov functions [5]. Along these lines, we describe δ-GAS-M$_q$ in terms of the existence of *incremental Lyapunov functions*.

Definition 4. *Define the* diagonal set Δ *as:* $\Delta = \{(x,x) \mid x \in \mathbb{R}^n\}$. *Consider a stochastic subsystem Σ_p and a continuous function $V_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ that is smooth on $\{\mathbb{R}^n \times \mathbb{R}^n\}\backslash\Delta$. Function V_p is called an incremental global asymptotic stability (δ-GAS-M$_q$) Lyapunov function in the* qth *mean for Σ_p, where $q \in \mathbb{N}$, if there exist $\mathcal{K}_\infty$ functions $\underline{\alpha}_p$, $\overline{\alpha}_p$, and a constant $\kappa_p \in \mathbb{R}^+$, such that*

(i) $\underline{\alpha}_p$ *(resp. $\overline{\alpha}_p$) is a convex (resp. concave) function;*
(ii) for any $x, x' \in \mathbb{R}^n$, $\underline{\alpha}_p\left(\|x - x'\|^q\right) \le V_p(x,x') \le \overline{\alpha}_p\left(\|x - x'\|^q\right)$;
(iii) for any $x, x' \in \mathbb{R}^n$, such that $x \neq x'$,

$$\mathcal{L}V_p(x,x') := [\partial_x V_p \;\; \partial_{x'} V_p] \begin{bmatrix} f_p(x) \\ f_p(x') \end{bmatrix} + \frac{1}{2} Tr\left(\begin{bmatrix} g_p(x) \\ g_p(x') \end{bmatrix} \begin{bmatrix} g_p^T(x) & g_p^T(x') \end{bmatrix} \begin{bmatrix} \partial_{x,x} V_p & \partial_{x,x'} V_p \\ \partial_{x',x} V_p & \partial_{x',x'} V_p \end{bmatrix} \right) \le -\kappa_p V_p(x,x').$$

The operator $\mathcal{L}$ is the infinitesimal generator associated to the stochastic subsystem (1) [24, Section 7.3], which characterizes the derivative of the expected value of functions of the process with respect to time. For non-probabilistic systems, $\mathcal{L}$ allows computing the conventional function derivative with respect to time. The symbols ∂_x and $\partial_{x,x'}$ denote first- and second-order partial derivatives with respect to x and x', respectively.

While condition (i) is not required in the context of non-probabilistic systems [5], the following theorem clarifies why such a condition is necessary for a stochastic subsystem, and describes δ-GAS-M$_q$ in terms of existence of a δ-GAS-M$_q$ Lyapunov function.

Theorem 1. *A stochastic subsystem Σ_p is δ-GAS-M$_q$ if it admits a δ-GAS-M$_q$ Lyapunov function.*

As qualitatively stated in the Introduction, it is known that a non-probabilistic switched system, whose subsystems are all δ-GAS, may exhibit some unstable behaviors under fast switching signals [15]. The same occurrence can affect stochastic switched systems endowed with δ-GAS-M$_q$ subsystems. The δ-GUAS property of non-probabilistic switched systems can be established by using a common (or global) Lyapunov function, or alternatively via multiple functions that are mode dependent [15]. This leads to the following extensions for δ-GUAS-M$_q$ of stochastic switched systems.

Assume that for any $p \in \mathsf{P}$, the stochastic subsystem Σ_p admits a δ-GAS-M$_q$ Lyapunov function V_p, satisfying conditions (i)-(iii) in Definition 4 with $\mathcal{K}_\infty$ functions $\underline{\alpha}_p$, $\overline{\alpha}_p$, and a constant $\kappa_p \in \mathbb{R}^+$. Let us introduce functions $\underline{\alpha}$ and $\overline{\alpha}$ and constant κ for use in the rest of the paper. Let the $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$, and the constant κ be defied as $\underline{\alpha} = \min\{\underline{\alpha}_1, \cdots, \underline{\alpha}_m\}$, $\overline{\alpha} = \max\{\overline{\alpha}_1, \cdots, \overline{\alpha}_m\}$, and $\kappa = \min\{\kappa_1, \cdots, \kappa_m\}$. First we show a result based on the existence of a common Lyapunov function, characterized by functions $\underline{\alpha} = \underline{\alpha}_1 = \cdots = \underline{\alpha}_m$ and $\overline{\alpha} = \overline{\alpha}_1 = \cdots = \overline{\alpha}_m$, and parameter κ.

Theorem 2. *Consider a stochastic switched system $\Sigma = (\mathbb{R}^n, \mathsf{P}, \mathcal{P}, F, G)$. If there exists a function V that is a common δ-GAS-M$_q$ Lyapunov function for all the subsystems $\{\Sigma_1, \cdots, \Sigma_m\}$, then Σ is δ-GUAS-M$_q$.*

The condition conservatively requires the existence of a single function V that is valid for all the subsystems Σ_p. When this common δ-GAS-M$_q$ Lyapunov function V fails to exist, the δ-GUAS-M$_q$ property of Σ can still be established by resorting to multiple δ-GAS-M$_q$ Lyapunov functions (one per mode) over a restricted set of switching signals. More precisely, from Definition 1 let $\mathcal{S}_{\tau_d}\left(\mathbb{R}_0^+, \mathsf{P}\right)$ denote the set of switching signals υ with *dwell time* $\tau_d \in \mathbb{R}_0^+$, meaning that $\upsilon \in \mathcal{S}\left(\mathbb{R}_0^+, \mathsf{P}\right)$ has dwell time τ_d if the switching times $t_1, t_2, \cdots$ (occurring at the discontinuity points of υ) satisfy $t_1 > \tau_d$ and $t_i - t_{i-1} \geq \tau_d$, for all $i \geq 2$. We now show a result based on multiple Lyapunov functions.

Theorem 3. *Let $\tau_d \in \mathbb{R}_0^+$, and consider a stochastic switched system $\Sigma_{\tau_d} = (\mathbb{R}^n, \mathsf{P}, \mathcal{P}_{\tau_d}, F, G)$ with $\mathcal{P}_{\tau_d} \subseteq \mathcal{S}_{\tau_d}\left(\mathbb{R}_0^+, \mathsf{P}\right)$. Assume that for any $p \in \mathsf{P}$, there exists a δ-GAS-M$_q$ Lyapunov function V_p for subsystem $\Sigma_{\tau_d,p}$ and that in addition there exits a constant $\mu \geq 1$ such that*

$$\forall x, x' \in \mathbb{R}^n, \;\; \forall p, p' \in \mathsf{P}, \;\; V_p(x, x') \leq \mu V_{p'}(x, x'). \tag{5}$$

If $\tau_d > \log \mu / \kappa$, then Σ_{τ_d} is δ-GUAS-M$_q$.

The above result can be practically interpreted as the following fact: global stability is preserved under subsystem stability and enough time spent in each location. Theorems 1, 2 and 3 provide sufficient conditions for certain stability properties, however they all hinge on finding proper Lyapunov functions. We look next into special instances where these functions are known explicitly or can be easily computed based on the model dynamics. The first result provides a sufficient condition for a particular function V_p to be a δ-GAS-M$_q$ Lyapunov function for a stochastic subsystem Σ_p, when $q = 1, 2$ (first or second mean).

Lemma 1. *Consider a stochastic subsystem Σ_p. Let $P_p \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix, and the function $V_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ be defined as follows:*

$$V_p(x, x') := \left(\widetilde{V}(x, x')\right)^{q/2} = \left(\frac{1}{q}\left(x - x'\right)^T P_p \left(x - x'\right)\right)^{q/2}, \tag{6}$$

and satisfy

$$(x - x')^T P_p(f_p(x) - f_p(x')) + \frac{1}{2}\left\|\sqrt{P_p}\left(g_p(x) - g_p(x')\right)\right\|_F^2 \leq -\kappa_p \left(V_p(x, x')\right)^{2/q}, \tag{7}$$

or, if f_p is differentiable, satisfy

$$(x - x')^T P_p \partial_x f_p(z)(x - x') + \frac{1}{2}\left\|\sqrt{P_p}\left(g_p(x) - g_p(x')\right)\right\|_F^2 \leq -\kappa_p \left(V_p(x, x')\right)^{2/q}, \tag{8}$$

for all x, x', z in $\mathbb{R}^n$, and for some constant $\kappa_p \in \mathbb{R}^+$. Then V_p is a δ-GAS-M$_q$ Lyapunov function for Σ_p, when $q \in \{1, 2\}$.

The next result provides a condition that is equivalent to (7) or to (8) for affine stochastic subsystems Σ_p (that is, for subsystems with affine drift and diffusion terms) in the form of a linear matrix inequality (LMI), which can be easily solved numerically.

Corollary 1. *Consider a stochastic subsystem Σ_p, where for any $x \in \mathbb{R}^n$ $f_p(x) := A_p x + b_p$ for some $A_p \in \mathbb{R}^{n\times n}, b_p \in \mathbb{R}^n$, and $g_p(x) := [\sigma_{1_p} x \;\; \sigma_{2_p} x \;\; \cdots \;\; \sigma_{q_p} x]$ for some $\sigma_{i_p} \in \mathbb{R}^{n\times n}$. Then, function V_p in* (6) *is a δ-GAS-M_q Lyapunov function for Σ_p if there exists a positive constant $\widehat{\kappa}_p \in \mathbb{R}^+$ satisfying the following LMI:*

$$P_p A_p + A_p^T P_p + \sum_{i=1}^{q} \sigma_{i_p}^T P_p \sigma_{i_p} \prec -\widehat{\kappa}_p P_p. \tag{9}$$

Notice that Corollary 1 allows obtaining tighter upper bounds for the inequalities (3) - (4) for any $p \in \mathsf{P}$, by selecting appropriate matrices P_p satisfying the LMI in (9).

4 Symbolic Models and Approximate Equivalence Relations

We employ the notion of system to provide an alternative description of stochastic switched systems that can be later directly related to their symbolic models.

Definition 5. *A system S is a tuple $S = (X, X_0, U, \longrightarrow, Y, H)$, where X is a set of states, $X_0 \subseteq X$ is a set of initial states, U is a set of inputs, $\longrightarrow \subseteq X \times U \times X$ is a transition relation, Y is a set of outputs, and $H : X \to Y$ is an output map.*

We write $x \xrightarrow{u} x'$ if $(x, u, x') \in \longrightarrow$. If $x \xrightarrow{u} x'$, we call state x' a u-successor, or simply a successor, of state x. For technical reasons, we assume that for each $x \in X$ and $u \in U$, there is some u-successor of x – let us remark that this is always the case for the considered systems later in this paper.

A system S is said to be

- *metric*, if the output set Y is equipped with a metric $\mathbf{d} : Y \times Y \to \mathbb{R}_0^+$;
- *finite*, if X is a finite set;
- *deterministic*, if for any state $x \in X$ and any input u, there exists exactly one u-successor.

For a system $S = (X, X_0, U, \longrightarrow, Y, H)$ and given any state $x_0 \in X_0$, a finite state run generated from x_0 is a finite sequence of transitions: $x_0 \xrightarrow{u_0} x_1 \xrightarrow{u_1} x_2 \xrightarrow{u_2} \cdots \xrightarrow{u_{n-2}} x_{n-1} \xrightarrow{u_{n-1}} x_n$, such that $x_i \xrightarrow{u_i} x_{i+1}$ for all $0 \leq i < n$. A finite state run can be trivially extended to an infinite state run as well.

We recall the notion of approximate (bi)simulation relation, introduced in [14], which is useful when analyzing or synthesizing controllers for deterministic systems.

Definition 6. *Let $S_a = (X_a, X_{a0}, U_a, \xrightarrow[a]{}, Y_a, H_a)$ and $S_b = (X_b, X_{b0}, U_b, \xrightarrow[b]{}, Y_b, H_b)$ be metric systems with the same output sets $Y_a = Y_b$ and metric $\mathbf{d}$. For $\varepsilon \in \mathbb{R}_0^+$, a relation $R \subseteq X_a \times X_b$ is said to be an ε-approximate simulation relation from S_a to S_b if the following three conditions are satisfied:*

(i) for every $x_{a0} \in X_{a0}$, there exists $x_{b0} \in X_{b0}$ with $(x_{a0}, x_{b0}) \in R$;
(ii) for every $(x_a, x_b) \in R$ we have $\mathbf{d}(H_a(x_a), H_b(x_b)) \leq \varepsilon$;
(iii) for every $(x_a, x_b) \in R$ we have that $x_a \xrightarrow[a]{u_a} x'_a$ in S_a implies the existence of $x_b \xrightarrow[b]{u_b} x'_b$ in S_b satisfying $(x'_a, x'_b) \in R$.

A relation $R \subseteq X_a \times X_b$ is said to be an ε-approximate bisimulation relation between S_a and S_b if R is an ε-approximate simulation relation from S_a to S_b and R^{-1} is an ε-approximate simulation relation from S_b to S_a.

System S_a is ε-approximately simulated by S_b, or S_b ε-approximately simulates S_a, denoted by $S_a \preceq_{\mathcal{S}}^{\varepsilon} S_b$, if there exists an ε-approximate simulation relation from S_a to S_b. System S_a is ε-approximate bisimilar to S_b, denoted by $S_a \cong_{\mathcal{S}}^{\varepsilon} S_b$, if there exists an ε-approximate bisimulation relation R between S_a and S_b.

Note that when $\varepsilon = 0$, the condition (ii) in the above definition is changed to $(x_a, x_b) \in R$ if and only if $H_a(x_a) = H_b(x_b)$, and R becomes an exact simulation relation, as introduced in [23]. Similarly, when $\varepsilon = 0$ and whenever applicable, R translates into an exact bisimulation relation.

5 Symbolic Models for Stochastic Switched Systems

This section contains the main contributions of this work. We show that for any stochastic switched system Σ (resp. Σ_{τ_d} as in Theorem 3), admitting a common (resp. multiple) δ-GAS-M$_q$ Lyapunov function(s), and for any precision level $\varepsilon \in \mathbb{R}^+$, we can construct a finite system that is ε-approximate bisimilar to Σ (resp. Σ_{τ_d}). In order to do so, we use systems as an abstract representation of stochastic switched systems, capturing all the information contained in them. More precisely, given a stochastic switched system $\Sigma = (\mathbb{R}^n, \mathsf{P}, \mathcal{P}, F, G)$, we define an associated metric system

$$S(\Sigma) = (X, X_0, U, \longrightarrow, Y, H),$$

where:

- X is the set of all $\mathbb{R}^n$-valued random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$;
- X_0 is the set of all $\mathbb{R}^n$-valued random variables that are measurable over the trivial sigma-algebra $\mathcal{F}_0$, i.e., the system starts from a deterministic initial condition, which is equivalently a random variable with a Dirac probability distribution;
- $U = \mathsf{P} \times \mathbb{R}^+$;
- $x \xrightarrow{p,\tau} x'$ if x and x' are measurable in $\mathcal{F}_t$ and $\mathcal{F}_{t+\tau}$, respectively, for some $t \in \mathbb{R}_0^+$, and there exists a solution process $\xi : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}^n$ of Σ satisfying $\xi(t) = x$ and $\xi_{xp}(\tau) = x'$ $\mathbb{P}$-a.s.;
- Y is the set of all $\mathbb{R}^n$-valued random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$;
- $H = 1_X$.

We assume that the output set Y is equipped with the natural metric $\mathbf{d}(y, y') = \mathbb{E}\left[\|y - y'\|^q\right]$, for any $y, y' \in Y$ and some $q \in \mathbb{N}$. Let us remark that the set of states of $S(\Sigma)$ is uncountable and that $S(\Sigma)$ is a deterministic system in the sense of Definition 5, since (cf. Subsection 2.2) its solution process is uniquely determined.

In subsequent developments, we will work with a sub-system of $S(\Sigma)$ obtained by restricting the transitions of $S(\Sigma)$ over times of the form $i\tau$, with $i \in \mathbb{N}_0$ and where τ is a given sampling time. This can be seen as a time discretization or a sampling of $S(\Sigma)$. This restriction is practically motivated by the fact that the switching in the original model Σ has to be controlled by a digital platform with a given clock period (τ). More precisely, given a stochastic switched system $\Sigma = (\mathbb{R}^n, \mathsf{P}, \mathcal{P}, F, G)$ and a sampling time $\tau \in \mathbb{R}^+$, we define the associated system $S_\tau(\Sigma) = \left(X_\tau, X_{\tau 0}, U_\tau, \underset{\tau}{\longrightarrow}, Y_\tau, H_\tau\right)$, where $X_\tau = X$, $X_{\tau 0} = X_0$, $U_\tau = \mathsf{P}$, $Y_\tau = Y$, $H_\tau = H$, and

– $x_\tau \xrightarrow[\tau]{p} x'_\tau$ if x_τ and x'_τ are measurable, respectively, in $\mathcal{F}_{k\tau}$ and $\mathcal{F}_{(k+1)\tau}$ for some $k \in \mathbb{N}_0$, and there exists a solution process $\xi : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}^n$ of Σ satisfying $\xi(k\tau) = x_\tau$ and $\xi_{x_\tau p}(\tau) = x'_\tau$ $\mathbb{P}$-a.s..

Note that a finite state run $x_0 \xrightarrow[\tau]{u_0} x_1 \xrightarrow[\tau]{u_1} \ldots \xrightarrow[\tau]{u_{N-1}} x_N$, of $S_\tau(\Sigma)$, where $u_i \in \mathsf{P}$ and $x_i = \xi_{x_{i-1}u_{i-1}}(\tau)$ for $i = 1, \cdots, N$, captures the trajectory of the stochastic switched system Σ at times $t = 0, \tau, \cdots, N\tau$, started from the deterministic initial condition x_0 and resulting from a switching signal υ obtained by the concatenation of the locations u_i $\big($i.e. $\upsilon(t) = u_{i-1}$ for any $t \in [(i-1)\tau, i\,\tau[\big)$, for $i = 1, \cdots, N$.

Before introducing the symbolic model for the stochastic switched system, we proceed with the following lemma, which provides an upper bound on the distance (in the 1st mean) between the solution processes of Σ_p and a corresponding non-probabilistic system obtained by disregarding the diffusion term (g_p).

Lemma 2. *[31] Consider a stochastic subsystem Σ_p such that $g_p(0_n) = 0_{n\times q}$. Suppose that the function V_p in* (6) *satisfies (7) or (8) for Σ_p. For any $x \in \mathbb{R}^n$ and any $p \in \mathsf{P}$, we have*

$$\mathbb{E}\left[\left\|\xi_{xp}(t) - \overline{\xi}_{xp}(t)\right\|\right] \leq h_p(t, g_p)\mathsf{e}^{-\kappa_p t},$$

where κ_p is the same constant introduced in (7) or (8), $\overline{\xi}_{xp}$ is the solution of the ordinary differential equation (ODE) $\dot{\overline{\xi}}_{xp}(t) = f_p\left(\overline{\xi}_{xp}(t)\right)$ starting from the initial condition x, and the nonnegative valued function h_p tends to zero as $t \to 0$ or as $\sup_x\{\|g_p(x)\|\} \to 0$. *Moreover, for g_p not identically equal to zero, h_p tends to $+\infty$ as $t \to +\infty$ with at most a linear rate.*

Similarly as done above, the following result shows that for a linear affine subsystem Σ_p (that is, with affine drift and linear diffusion terms) function $h_p(t, g_p)$ has an explicit form.

Corollary 2. *[31] Consider a stochastic subsystem Σ_p, where for any $x \in \mathbb{R}^n$ $f_p(x) := A_p x + b_p$, for some $A_p \in \mathbb{R}^{n\times n}, b_p \in \mathbb{R}^n$, and $g_p(x) :=$ $[\sigma_{1_p}x \;\; \sigma_{2_p}x \;\; \cdots \;\; \sigma_{q_p}x]$, for some $\sigma_{i_p} \in \mathbb{R}^{n\times n}$. If the dynamics of Σ_p are restricted to a compact subset $D_p \subset \mathbb{R}^n$, then we obtain*

$$h_p(t, g_p) =$$
$$\sqrt{\frac{n\lambda_{\max}\left(\sum_{i=1}^q \sigma_{i_p}^T P_p \sigma_{i_p}\right)}{\lambda_{\min}(P_p)}} \cdot \sqrt{\int_0^t \left(\|\mathsf{e}^{A_p s}\| \sup_{x\in D_p}\{\|x\|\} + \int_0^s \|\mathsf{e}^{A_p r} b_p\|\, dr\right)^2 ds}\,\mathsf{e}^{-\kappa_p t}.$$

For later use, we introduce function $h(t, G) = \max\{h_1(t, g_1), \cdots, h_m(t, g_m)\}$ for all $t \in \mathbb{R}_0^+$.

In order to show the main results, we raise the following supplementary assumption on the δ-GAS-M$_q$ Lyapunov functions V_p: for all $p \in \mathsf{P}$, there exists a $\mathcal{K}_\infty$ and concave function $\widehat{\gamma}_p$ such that

$$|V_p(x, y) - V_p(x, z)| \leq \widehat{\gamma}_p(\|y - z\|), \tag{10}$$

for any $x, y, z \in \mathbb{R}^n$. This assumption is not restrictive, provided we are interested in the dynamics of Σ on a compact subset $D \subset \mathbb{R}^n$, which is often the case in practice.

For all $x, y, z \in D$, by applying the mean value theorem to the function $y \to V_p(x, y)$, one gets

$$|V_p(x,y) - V_p(x,z)| \le \widehat{\gamma}_p\left(\|y-z\|\right), \text{ where } \widehat{\gamma}_p(r) = \left(\max_{(x,y)\in D\setminus\Delta}\left\|\frac{\partial V_p(x,y)}{\partial y}\right\|\right) r.$$

In particular, for the δ-GAS-M$_1$ Lyapunov function V_p defined in (6), we obtain $\widehat{\gamma}_p(r) = \frac{\lambda_{\max}(P_p)}{\sqrt{\lambda_{\min}(P_p)}} r$ [29, Proposition 10.5]. For later use, let us define the $\mathcal{K}_\infty$ function $\widehat{\gamma}$ such that $\widehat{\gamma} = \max\{\widehat{\gamma}_1, \cdots, \widehat{\gamma}_m\}$. (Note that, for the case of a common Lyapunov function, we have: $\widehat{\gamma} = \widehat{\gamma}_1 = \cdots = \widehat{\gamma}_m$.) We proceed presenting the main results of this work.

5.1 Common Lyapunov function

We first show a results based on the existence of a common δ-GAS-M$_q$ Lyapunov function for subsystems $\Sigma_1, \cdots, \Sigma_m$. Consider a stochastic switched system $\Sigma = (\mathbb{R}^n, \mathsf{P}, \mathcal{P}, F, G)$, and a pair $\mathsf{q} = (\tau, \eta)$ of quantization parameters, where τ is the sampling time and η is the state space quantization. Given Σ and q, consider the following system:

$$S_{\mathsf{q}}(\Sigma) = (X_{\mathsf{q}}, X_{\mathsf{q}0}, U_{\mathsf{q}}, \underset{\mathsf{q}}{\longrightarrow}, Y_{\mathsf{q}}, H_{\mathsf{q}}), \tag{11}$$

where $X_{\mathsf{q}} = [\mathbb{R}^n]_\eta$, $X_{\mathsf{q}0} = [\mathbb{R}^n]_\eta$, $U_{\mathsf{q}} = \mathsf{P}$, and

- $x_{\mathsf{q}} \xrightarrow[\mathsf{q}]{p} x'_{\mathsf{q}}$ if $\left\|\overline{\xi}_{x_{\mathsf{q}}p}(\tau) - x'_{\mathsf{q}}\right\| \le \eta$, where $\dot{\overline{\xi}}_{x_{\mathsf{q}}p}(t) = f_p\left(\overline{\xi}_{x_{\mathsf{q}}p}(t)\right)$;
- Y_{q} is the set of all $\mathbb{R}^n$-valued random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$;
- $H_{\mathsf{q}} = \imath : X_{\mathsf{q}} \hookrightarrow Y_{\mathsf{q}}$.

In order to relate models, the output set Y_{q} is taken to be that of the stochastic switched system $S_\tau(\Sigma)$. Therefore, in the definition of H_{q}, the inclusion map $\imath$ is meant, with a slight abuse of notation, as a mapping from a deterministic grid point to a random variable with a Dirac probability distribution centered at the grid point. As argued in [29], there is no loss of generality to alternatively assume that $Y_{\mathsf{q}} = X_{\mathsf{q}}$ and $H_{\mathsf{q}} = 1_{X_{\mathsf{q}}}$.

The transition relation of $S_{\mathsf{q}}(\Sigma)$ is well defined in the sense that for every $x_{\mathsf{q}} \in [\mathbb{R}^n]_\eta$ and every $p \in \mathsf{P}$ there always exists $x'_{\mathsf{q}} \in [\mathbb{R}^n]_\eta$ such that $x_{\mathsf{q}} \xrightarrow[\mathsf{q}]{p} x'_{\mathsf{q}}$. This can be seen since by definition of $[\mathbb{R}^n]_\eta$, for any $\widehat{x} \in \mathbb{R}^n$ there always exists a state $\widehat{x}' \in [\mathbb{R}^n]_\eta$ such that $\|\widehat{x} - \widehat{x}'\| \le \eta$. Hence, for $\overline{\xi}_{x_{\mathsf{q}}p}(\tau)$ there always exists a state $x'_{\mathsf{q}} \in [\mathbb{R}^n]_\eta$ satisfying $\left\|\overline{\xi}_{x_{\mathsf{q}}p}(\tau) - x'_{\mathsf{q}}\right\| \le \eta$.

We can now present one of the main results of the paper, which relates the existence of a common δ-GAS-M$_q$ Lyapunov function for the subsystems $\Sigma_1, \cdots, \Sigma_m$ to the construction of a finite symbolic model that is approximately bisimilar to the original system.

Theorem 4. *Let $\Sigma = (\mathbb{R}^n, \mathsf{P}, \mathcal{P}, F, G)$ be a stochastic switched system admitting a common δ-GAS-M$_q$ Lyapunov function V, as defined in Lemma 1, for subsystems $\Sigma_1, \cdots, \Sigma_m$, when $q \in \{1, 2\}$. Moreover, assume that V satisfies (10) for some $\mathcal{K}_\infty$ function $\widehat{\gamma}$. For any $\varepsilon \in \mathbb{R}^+$, and any double $\mathsf{q} = (\tau, \eta)$ of quantization parameters satisfying*

$$\overline{\alpha}\left(\eta^q\right) \le \underline{\alpha}(\varepsilon), \tag{12}$$

$$\mathsf{e}^{-\kappa\tau}\underline{\alpha}(\varepsilon) + \widehat{\gamma}\left(h(\tau, G)\mathsf{e}^{-\kappa\tau} + \eta\right) \le \underline{\alpha}(\varepsilon), \tag{13}$$

we have that $S_{\mathsf{q}}(\Sigma) \cong^{\varepsilon}_{\mathcal{S}} S_\tau(\Sigma)$.

Recall that $S_\tau(\Sigma)$ denotes an alternative (sampled) representation of the model Σ. It can be readily seen that when we are interested in the dynamics of Σ on a compact subset $D \subset \mathbb{R}^n$ and for a given precision ε, there always exists a sufficiently large value of τ and a small value of η such that $\eta \leq span(D)$ and the conditions in (12) and (13) are satisfied. For a given fixed sampling time τ, the precision ε is lower bounded by:

$$\varepsilon > \underline{\alpha}^{-1}\left(\frac{\widehat{\gamma}\left(h(\tau, G)\mathsf{e}^{-\kappa\tau}\right)}{1-\mathsf{e}^{-\kappa\tau}}\right). \tag{14}$$

One can easily verify that the lower bound on ε in (14) goes to zero as τ goes to infinity. Furthermore, one can try to minimize the lower bound on ε in (14) by appropriately choosing matrices P_p in (6), for any $p \in \mathsf{P}$.

Note that the results in [15, Theorem 4.1] for non-probabilistic models are fully recovered by the statement in Theorem 4 if the stochastic switched system Σ is not affected by any noise, implying that $h_p(t, g_p)$ is identically zero for all $p \in \mathsf{P}$, and that the δ-GAS-M$_q$ common Lyapunov function simply reduces to being δ-GAS. As a side remark, we are currently investigating the extension of Theorem 4 to the cases where the common Lyapunov function V is not necessarily of the form of (6).

5.2 Multiple Lyapunov functions

If a common δ-GAS-M$_q$ Lyapunov function does not exist, one can still attempt computing approximately bisimilar symbolic models by seeking location-dependent Lyapunov functions and by restricting the set of switching signals using a dwell time τ_d. For simplicity and without loss of generality, we assume that τ_d is an integer multiple of τ, i.e. that there exists $N \in \mathbb{N}$ such that $\tau_d = N\tau$.

Given a stochastic switched system $\Sigma_{\tau_d} = (\mathbb{R}^n, \mathsf{P}, \mathcal{P}_{\tau_d}, F, G)$ and a sampling time $\tau \in \mathbb{R}^+$, we define the system

$$S_\tau\left(\Sigma_{\tau_d}\right) = (X_\tau, X_{\tau 0}, U_\tau, \underset{\tau}{\longrightarrow}, Y_\tau, H_\tau),$$

where:

- $X_\tau = \mathcal{X} \times \mathsf{P} \times \{1, \cdots, N-1\}$, where $\mathcal{X}$ is the set of all $\mathbb{R}^n$-valued random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$;
- $X_{\tau 0} = \mathcal{X}_0 \times \mathsf{P} \times \{0\}$, where $\mathcal{X}_0$ is the set of all $\mathbb{R}^n$-valued random variables that are measurable with respect to the trivial sigma-algebra $\mathcal{F}_0$, i.e., the stochastic switched system starts from a deterministic initial condition;
- $U_\tau = \mathsf{P}$;
- $(x_\tau, p, i) \underset{\tau}{\overset{p}{\longrightarrow}} (x'_\tau, p', i')$ if x_τ and x'_τ are measurable, respectively, in $\mathcal{F}_{k\tau}$ and $\mathcal{F}_{(k+1)\tau}$ for some $k \in \mathbb{N}_0$, and there exists a solution process $\xi : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}^n$ of Σ satisfying $\xi(k\tau) = x_\tau$ and $\xi_{x_\tau p}(\tau) = x'_\tau$ $\mathbb{P}$-a.s. and one of the following holds:
  - $i < N-1$, $p' = p$, and $i' = i+1$: switching is not allowed because the time elapsed since the latest switch is strictly smaller than the dwell time;
  - $i = N-1$, $p' = p$, and $i' = N-1$: switching is allowed but no location switch occurs;
  - $i = N-1$, $p' \neq p$, and $i' = 0$: switching is allowed and a location switch occurs.
- $Y_\tau = \mathcal{X}$ is the set of all $\mathbb{R}^n$-valued random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$;
- H_τ is the map taking $(x_\tau, p, i) \in \mathcal{X} \times \mathsf{P} \times \{1, \cdots, N-1\}$ to $x_\tau \in \mathcal{X}$.

We assume that the output set Y_τ is equipped with the natural metric $\mathbf{d}(y,y') = \mathbb{E}\left[\|y-y'\|^q\right]$, for any $y,y' \in Y_\tau$ and some $q \in \mathbb{N}$. One can readily verify that the output trajectories of $S_\tau\left(\Sigma_{\tau_d}\right)$ are the output trajectories of $S_\tau(\Sigma)$ corresponding to switching signals with dwell time $\tau_d = N\tau$.

Consider a stochastic switched system $\Sigma_{\tau_d} = (\mathbb{R}^n, \mathsf{P}, \mathcal{P}_{\tau_d}, F, G)$ and a pair $\mathsf{q} = (\tau,\eta)$ of quantization parameters, where τ is the sampling time and η is the state space quantization. Given Σ_{τ_d} and q, consider the following system:

$$S_{\mathsf{q}}\left(\Sigma_{\tau_d}\right) = (X_{\mathsf{q}}, X_{\mathsf{q}0}, U_{\mathsf{q}}, \underset{\mathsf{q}}{\longrightarrow}, Y_{\mathsf{q}}, H_{\mathsf{q}}), \tag{15}$$

where $X_{\mathsf{q}} = [\mathbb{R}^n]_\eta \times \mathsf{P} \times \{0,\cdots,N-1\}$, $X_{\mathsf{q}0} = [\mathbb{R}^n]_\eta \times \mathsf{P} \times \{0\}$, $U_{\mathsf{q}} = \mathsf{P}$, and

- $(x_{\mathsf{q}},p,i) \overset{p}{\underset{\mathsf{q}}{\longrightarrow}} \left(x'_{\mathsf{q}},p',i'\right)$ if $\left\|\overline{\xi}_{x_{\mathsf{q}}p}(\tau) - x'_{\mathsf{q}}\right\| \leq \eta$, where $\dot{\overline{\xi}}_{x_{\mathsf{q}}p}(t) = f_p\left(\overline{\xi}_{x_{\mathsf{q}}p}(t)\right)$ and one of the following holds:
  - $i < N-1$, $p' = p$, and $i' = i+1$;
  - $i = N-1$, $p' = p$, and $i' = N-1$;
  - $i = N-1$, $p' \neq p$, and $i' = 0$.
- $Y_{\mathsf{q}} = \mathcal{X}$ is the set of all $\mathbb{R}^n$-valued random variables defined on $(\Omega,\mathcal{F},\mathbb{P})$;
- H_{q} is the map taking $(x_{\mathsf{q}},p,i) \in [\mathbb{R}^n]_\eta \times \mathsf{P} \times \{1,\cdots,N-1\}$ to a random variable with a Dirac probability distribution centered at x_{q}.

Similar to what we showed in the case of a common Lyapunov function, the transition relation of system $S_{\mathsf{q}}\left(\Sigma_{\tau_d}\right)$ is well defined in the sense that for every $(x_{\mathsf{q}},p,i) \in [\mathbb{R}^n]_\eta \times \mathsf{P} \times \{0,\cdots,N-1\}$ there always exists $\left(x'_{\mathsf{q}},p',i'\right) \in [\mathbb{R}^n]_\eta \times \mathsf{P} \times \{0,\cdots,N-1\}$ such that $(x_{\mathsf{q}},p,i) \overset{p}{\underset{\mathsf{q}}{\longrightarrow}} \left(x'_{\mathsf{q}},p',i'\right)$.

We present the second main result of the paper, which relates the existence of multiple Lyapunov functions for a stochastic switched system to that of a symbolic model.

Theorem 5. *Consider $\tau_d \in \mathbb{R}_0^+$, and a stochastic switched system $\Sigma_{\tau_d} = (\mathbb{R}^n, \mathsf{P}, \mathcal{P}_{\tau_d}, F, G)$ such that $\tau_d = N\tau$, for some $N \in \mathbb{N}$. Let us assume that for all $p \in \mathsf{P}$, there exists a δ-GAS-M$_q$ Lyapunov function V_p for subsystem $\Sigma_{\tau_d,p}$, as defined in Lemma 1, when $q \in \{1,2\}$. Moreover, assume that (5) and (10) hold for some $\mu \geq 1$ and concave $\mathcal{K}_\infty$ functions $\widehat{\gamma}_1,\cdots,\widehat{\gamma}_m$. If $\tau_d > \log\mu/\kappa$, for any $\varepsilon \in \mathbb{R}^+$, and any pair $\mathsf{q} = (\tau,\eta)$ of quantization parameters satisfying*

$$\overline{\alpha}\left(\eta^q\right) \leq \underline{\alpha}(\varepsilon), \tag{16}$$

$$\widehat{\gamma}\left(h(\tau,G)\mathsf{e}^{-\kappa\tau} + \eta\right) \leq \frac{\frac{1}{\mu} - \mathsf{e}^{-\kappa\tau_d}}{1 - \mathsf{e}^{-\kappa\tau_d}}\left(1 - \mathsf{e}^{-\kappa\tau}\right)\underline{\alpha}(\varepsilon), \tag{17}$$

we have that $S_{\mathsf{q}}\left(\Sigma_{\tau_d}\right) \cong_{\mathcal{S}}^{\varepsilon} S_\tau\left(\Sigma_{\tau_d}\right)$.

It can be readily seen that when we are interested in the dynamics of Σ_{τ_d} on a compact subset $D \subset \mathbb{R}^n$ and for a precision ε, there always exists sufficiently large value of τ and small value of η such that $\eta \leq span(D)$ and the conditions in (16) and (17) are satisfied. For a given fixed sampling time τ, the precision ε is lower bounded by:

$$\varepsilon \geq \underline{\alpha}^{-1}\left(\frac{\widehat{\gamma}\left(h(\tau,G)\mathsf{e}^{-\kappa\tau}\right)}{1 - \mathsf{e}^{-\kappa\tau}} \cdot \frac{1 - \mathsf{e}^{-\kappa\tau_d}}{\frac{1}{\mu} - \mathsf{e}^{-\kappa\tau_d}}\right). \tag{18}$$

The properties of the bound in (18) are analogous to those of the case of a common Lyapunov function. Similarly, Theorem 5 subsumes [15, Theorem 4.2] over non-probabilistic models.

Symbolic models are prone to be easily model checked or employed towards controller synthesis. It is of interest to understand how abstract controllers can be used over the concrete models. The next proposition elucidates how a controller S_{cont} synthesized to solve a simulation game over $S_{\mathsf{q}}(\Sigma)$ (resp. $S_{\mathsf{q}}(\Sigma_{\tau_d})$) can be refined to a controller for $S_\tau(\Sigma)$ (resp. $S_\tau\left(\Sigma_{\tau_d}\right)$). A detailed description of the feedback composition (denoted by $\|$) and of its properties for metric systems can be found in [29].

Proposition 1. *[29] Consider a stochastic switched system Σ $(resp.\, \Sigma_{\tau_d})$, and a specification described by a deterministic system $S_{spec} = \left(X_{spec}, X_{spec0}, U_{spec}, \underset{spec}{\longrightarrow}, Y_{spec}, H_{spec}\right)$, where X_{spec} is a finite subset of $\mathbb{R}^n$, $X_{spec0} \subseteq X_{spec}$, $U_{spec} = \{u_{spec}\}$, $\underset{spec}{\longrightarrow} \subseteq X_{spec} \times U_{spec} \times X_{spec}$, Y_{spec} is the set of all $\mathbb{R}^n$-valued random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $H_{spec} = \imath : X_{spec} \hookrightarrow Y_{spec}$. Assume that $S_\tau(\Sigma) \cong^{\varepsilon}_{\mathcal{S}} S_{\mathsf{q}}(\Sigma)$ and S_{cont} is synthesized to solve exactly a simulation game for $S_{\mathsf{q}}(\cdot)$ and a specification S_{spec}: $S_{cont} \parallel S_{\mathsf{q}} \preceq^{0}_{\mathcal{S}} S_{spec}$ (resp. $S_{cont} \parallel S_{\mathsf{q}} \cong^{0}_{\mathcal{S}} S_{spec}$). Then, using $S'_{cont} = S_{cont} \parallel S_{\mathsf{q}}$ as a controller for $S_\tau(\cdot)$, we obtain: $S'_{cont} \parallel S_\tau \preceq^{\varepsilon}_{\mathcal{S}} S_{spec}$ (resp. $S'_{cont} \parallel S_\tau \cong^{\varepsilon}_{\mathcal{S}} S_{spec}$).*

Remark 1 (Relationship with notions in the literature). The approximate bisimulation notion in Definition 6 is structurally different than the probabilistic version discussed for finite state, *discrete-time* labeled Markov chains in [11], which can also be extended to continuous-space processes as in [10]. The notion in this work can be instead related to the approximate probabilistic bisimulation notion discussed in [1], which lower bounds the probability that the Euclidean distance between abstract and concrete models remains close over a given time horizon: both notions hinge on distances over trajectories, rather than over transition probabilities as in [10, 11]. As a first step in this direction, the authors are working on establishing a probabilistic approximate bisimulation relation between $S_\tau(\Sigma)$ (resp. $S_\tau\left(\Sigma_{\tau_d}\right)$) and $S_{\mathsf{q}}(\Sigma)$ (resp. $S_{\mathsf{q}}\left(\Sigma_{\tau_d}\right)$) point-wise in time [31, Lemma 5.8]: this result is sufficient to work with LTL specifications that need to be satisfied at a single time instance, such as next ($\bigcirc$) and reach ($\Diamond$).

6 Case Study

We experimentally demonstrate the effectiveness of the results. In the example below, the computation of the abstractions $S_{\mathsf{q}}(\Sigma)$ has been performed via the software tool Pessoa [22] on a laptop with CPU 2GHz Intel Core i7. Controllers enforcing the specification were found by using standard algorithms from game theory [20, 30], as implemented in Pessoa. The terms $W^i_t, i = 1, 2,$ denote the standard Brownian motion.

Σ is a simple thermal model of a two-room building, borrowed from [8, 13], affected by noise and described by the following stochastic differential equations:

$$\begin{cases} \mathrm{d}\,\xi_1 = \left(\alpha_{21}\left(\xi_2 - \xi_1\right) + \alpha_{e1}\left(T_e - \xi_1\right) + \alpha_f\left(T_f - \xi_1\right)(p-1)\right)\mathrm{d}\,t + \sigma_1\xi_1\,\mathrm{d}\,W^1_t, \\ \mathrm{d}\,\xi_2 = \left(\alpha_{12}\left(\xi_1 - \xi_2\right) + \alpha_{e2}\left(T_e - \xi_2\right)\right)\mathrm{d}\,t + \sigma_2\xi_2\,\mathrm{d}\,W^2_t, \end{cases} \tag{19}$$

where ξ_1 and ξ_2 denote the temperature in each room, $T_e = 10$ (degrees Celsius) is the external temperature and $T_f = 50$ is the temperature of a heater that can be switched off ($p = 1$) or on ($p = 2$): these two operations correspond to the locations P of the model, whereas the state space is $\mathbb{R}^2$. The drifts f_p and diffusion terms $g_p, p = 1, 2,$ can be simply written out of (19) and are affine. The parameters of the drifts are chosen based on the ones in [13] as follows: $\alpha_{21} = \alpha_{12} = 5\times10^{-2}$, $\alpha_{e1} = 5\times10^{-3}$, $\alpha_{e2} = 3.3\times10^{-3}$, and $\alpha_f = 8.3\times10^{-3}$. The noise parameters $\sigma_1 = 0.001$, and $\sigma_2 = 0.001$ are chosen to be similar to those in [8]. We work on the subset $D = [20,\ 22] \times [20,\ 22] \subset \mathbb{R}^2$ of the state space of Σ. Within D one can conservatively overapproximate the multiplicative noises in (19) as additive noises with variance between 0.02 and 0.022.

It can be readily verified that the function $V(x_1, x_2) = \sqrt{(x_1 - x_2)^T(x_1 - x_2)}$ is a common δ-GAS-M$_1$ Lyapunov function for Σ, satisfying the LMI condition (9) with $P_p = I_2$, and $\widehat{\kappa}_p = 0.0083$, for $p \in \{1, 2\}$.

For a given sampling time $\tau = 20$ time units, using inequality (14) the precision ε is lower bounded by the quantity 1.09. While one can reduce this lower bound by increasing the sampling time, as discussed later the empirical bound computed in the experiments is significantly lower than the theoretical bound $\varepsilon = 1.09$. For a selected precision $\varepsilon = 1.1$, the discretization parameter η for $S_{\mathsf{q}}(\Sigma)$, computed from Theorem 4, equals to 0.003. This has lead to a symbolic system $S_{\mathsf{q}}(\Sigma)$ with a resulting number of states equal to 895122. The CPU time employed to compute the abstraction amounted to 506.32 seconds.

Consider the objective to design a controller (switching policy) forcing the first moment of the trajectories of Σ to stay within D. This objective can be encoded via the LTL specification $\Box D$. Furthermore, to add an additional discrete component to the problem, we assume that the heater has to stay in the off mode ($p = 1$) at most one time slot every two slots. A time slot is an interval of the form $[k\tau, (k + 1)\tau[$, with $k \in \mathbb{N}$ and where τ is the sampling time. Possible switching policies are for instance:

$$|12|12|12|12|12|12|12|\cdots, \ |21|21|21|21|21|21|21|\cdots, \ |12|21|22|12|12|21|22|\cdots,$$

where 2 denotes a slot where the heater is on ($p = 2$) and 1 denotes a slot where heater is off ($p = 1$). This constraint on the switching policies can be represented by the finite system (labeled automaton) in Figure 1, where the allowed initial states are distinguished as targets of a sourceless arrow. The CPU time for synthesizing the controller amounted to 21.14 seconds. In Figure 2, we show several realizations of closed-loop trajectory $\xi_{x_0\upsilon}$ stemming from initial condition $x_0 = (21, 21)$ (left panel), as well as the corresponding evolution of switching signal υ (right panel), where the finite system is initialized in state q_1. Furthermore, in Figure 2 (middle panels), we show the average value over 100 experiments of the distance in time of the solution process $\xi_{x_0\upsilon}$ to the set D, namely $\|\xi_{x_0\upsilon}(t)\|_D$, where the point-to-set distance is defined as $\|x\|_D = \inf_{d\in D}\|x - d\|$. Notice that the average distance is significantly lower than the precision $\varepsilon = 1.1$, as expected since the conditions based on Lyapunov functions can lead to conservative bounds. (As discussed in Corollary 1, bounds can be improved by seeking optimized Lyapunov functions.)

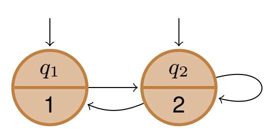

Fig. 1: Finite system describing the constraint over the switching policies. The lower part of the states are labeled with the outputs (2 and 1) denoting whether heater is on ($p = 2$) or off ($p = 1$).

A second example using multiple Lyapunov functions is provided in the Appendix.

7 Conclusions and Future Work

This work has shown that any stochastic switched system Σ (resp. Σ_{τ_d}), admitting a common (multiple) δ-GAS-M$_q$ Lyapunov function(s), as in (6), and evolving within a

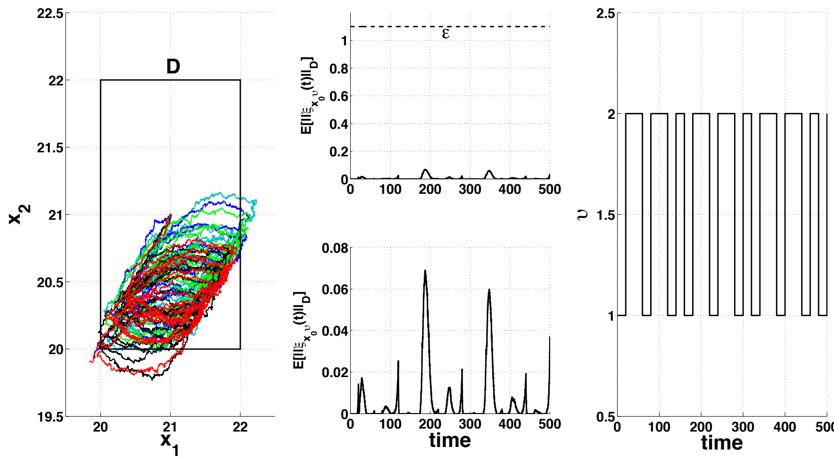

Fig. 2: Several realizations of the closed-loop trajectory $\xi_{x_0 \upsilon}$ with initial condition $x_0 = (21, 21)$ (left panel). Average values (over 100 experiments) of the distance of the solution process $\xi_{x_0 \upsilon}$ to the set D, in different vertical scales (middle panels). Evolution of the synthesized switching signal υ (right panel), where the finite system initialized from state q_1.

compact set of states, admits a finite approximately bisimilar symbolic model $S_{\mathsf{q}}(\Sigma)$ ($S_{\mathsf{q}}(\Sigma_{\tau_d})$). The constructed symbolic model can be used to synthesize controllers enforcing complex logic specifications, expressed via linear temporal logic or as automata on infinite strings.

The main limitation of the design methodology developed in this paper lies in the cardinality of the set of states of the computed symbolic model, which relates to the continuous dimension of the concrete system: the authors are currently investigating several different techniques to address this limitation. Furthermore, the authors are currently working toward extensions of the results over general stochastic hybrid systems.

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8 Appendix

Proof (of Theorem 1). The proof is a consequence of the application of Gronwall's inequality and of Ito's lemma [24, p. 80 and 123]. Assume that there exists a δ-GAS-M$_q$ Lyapunov function in the sense of Definition 4. For any $t \in \mathbb{R}_0^+$, and any $\mathbb{R}^n$-valued random variables a and a' that are measurable in $\mathcal{F}_0$, we obtain

$$\begin{aligned}\mathbb{E}\left[V_p(\xi_{ap}(t), \xi_{a'p}(t))\right] &= \mathbb{E}\left[V_p(a, a') + \int_0^t \mathcal{L}V_p(\xi_{ap}(s), \xi_{a'p}(s))ds\right] \\ &\leq \mathbb{E}\left[V_p(a, a') + \int_0^t \left(-\kappa_p V_p(\xi_{ap}(s), \xi_{a'p}(s))\right) ds\right] \\ &\leq -\kappa_p \int_0^t \mathbb{E}\left[V_p(\xi_{ap}(s), \xi_{a'p}(s))\right] ds + \mathbb{E}\left[V_p(a, a')\right],\end{aligned}$$

which, by virtue of Gronwall's inequality, leads to

$$\mathbb{E}\left[V_p(\xi_{ap}(t), \xi_{a'p}(t))\right] \leq \mathbb{E}[V_p(a, a')]\mathsf{e}^{-\kappa_p t}.$$

Hence, using property (ii) in Definition 4, we have

$$\begin{aligned}&\underline{\alpha}_p\left(\mathbb{E}\left[\|\xi_{ap}(t) - \xi_{a'p}(t)\|^q\right]\right) \leq \mathbb{E}\left[\underline{\alpha}_p\left(\|\xi_{ap}(t) - \xi_{a'p}(t)\|^q\right)\right] \leq \mathbb{E}\left[V_p\left(\xi_{ap}(t), \xi_{a'p}(t)\right)\right] \\ &\leq \mathbb{E}\left[V_p(a, a')\right]\mathsf{e}^{-\kappa_p t} \leq \mathbb{E}\left[\overline{\alpha}_p\left(\left\|a - a'\right\|^q\right)\right]\mathsf{e}^{-\kappa_p t} \leq \overline{\alpha}_p\left(\mathbb{E}\left[\left\|a - a'\right\|^q\right]\right)\mathsf{e}^{-\kappa_p t},\end{aligned}$$

where the first and last inequalities follow from property (i) and Jensen's inequality [24, p. 310]. Since $\underline{\alpha}_p \in \mathcal{K}_\infty$, we obtain

$$\mathbb{E}\left[\|\xi_{ap}(t) - \xi_{a'p}(t)\|^q\right] \leq \underline{\alpha}_p^{-1}\left(\overline{\alpha}_p\left(\mathbb{E}\left[\|a - a'\|^q\right]\right)\mathsf{e}^{-\kappa_p t}\right).$$

Therefore, by introducing function β_p as

$$\beta_p(r, s) := \underline{\alpha}_p^{-1}\left(\overline{\alpha}_p(r)\,\mathsf{e}^{-\kappa_p s}\right),$$

condition (3) is satisfied. Hence, the stochastic subsystem Σ_p is δ-GAS-M$_q$. □

Proof (of Theorem 2). The proof is a consequence of the application of Gronwall's inequality and of Ito's lemma [24, p. 80 and 123]. For any $\mathbb{R}^n$-valued random variables a and a' that are measurable in $\mathcal{F}_0$, any switching signal $\upsilon \in \mathcal{P}$, and for all $t \in \mathbb{R}_0^+$ where υ is continuous, we have

$$\mathcal{L}V\left(\xi_{a\upsilon}(t), \xi_{a'\upsilon}(t)\right) \leq -\kappa V(\xi_{a\upsilon}(t), \xi_{a'\upsilon}(t)).$$

By using continuity of V and for all $t \in \mathbb{R}_0^+$, one gets

$$\begin{aligned}\mathbb{E}\left[V(\xi_{a\upsilon}(t), \xi_{a'\upsilon}(t))\right] &\leq \mathbb{E}\left[V(a, a') + \int_0^t \left(-\kappa V(\xi_{a\upsilon}(s), \xi_{a'\upsilon}(s))\right) ds\right] \\ &\leq -\kappa \int_0^t \mathbb{E}\left[V(\xi_{a\upsilon}(s), \xi_{a'\upsilon}(s))\right] ds + \mathbb{E}\left[V(a, a')\right],\end{aligned}$$

which, by virtue of Gronwall's inequality, leads to

$$\mathbb{E}\left[V(\xi_{a\upsilon}(t), \xi_{a'\upsilon}(t))\right] \leq \mathbb{E}[V(a, a')]\mathsf{e}^{-\kappa t}.$$

Since here $\mathcal{K}_\infty$ functions $\underline{\alpha}$ and $\overline{\alpha}$ are convex and concave, respectively, and using Jensen's inequality, we have

$$\begin{aligned}&\underline{\alpha}\left(\mathbb{E}\left[\|\xi_{av}(t)-\xi_{a'v}(t)\|^q\right]\right)\leq\mathbb{E}\left[\underline{\alpha}\left(\|\xi_{av}(t)-\xi_{a'v}(t)\|^q\right)\right]\leq\mathbb{E}\left[V\left(\xi_{av}(t),\xi_{a'v}(t)\right)\right]\\&\leq\mathbb{E}\left[V(a,a')\right]\mathsf{e}^{-\kappa t}\leq\mathbb{E}\left[\overline{\alpha}\left(\|a-a'\|^q\right)\right]\mathsf{e}^{-\kappa t}\leq\overline{\alpha}\left(\mathbb{E}\left[\|a-a'\|^q\right]\right)\mathsf{e}^{-\kappa t}.\end{aligned}$$

Since $\underline{\alpha}\in\mathcal{K}_\infty$, we obtain

$$\mathbb{E}\left[\|\xi_{av}(t)-\xi_{a'v}(t)\|^q\right]\leq\underline{\alpha}^{-1}\left(\overline{\alpha}\left(\mathbb{E}\left[\|a-a'\|^q\right]\right)\mathsf{e}^{-\kappa t}\right),$$

for all $t\in\mathbb{R}_0^+$. Then condition (4) holds with the function $\beta(r,s)=\underline{\alpha}^{-1}\left(\overline{\alpha}(r)\mathsf{e}^{-\kappa s}\right)$.

□

Proof (of Theorem 3). The proof was inspired by the proof of Theorem 2.8 in [15]. We show the result for the case that switching signals have infinite number of discontinuities (switching times). A proof for the case of finite discontinuities can be written in a similar way. Let a and a' be any $\mathbb{R}^n$-valued random variables that are measurable in $\mathcal{F}_0$, $v\in\mathcal{P}_{\tau_d}$, $t_0=0$, and let $p_{i+1}\in\mathsf{P}$ denotes the value of the switching signal on the open interval $(t_i,\ t_{i+1})$, for $i\in\mathbb{N}_0$. Using (iii) in Definition 4 for all $i\in\mathbb{N}_0$ and $t\in(t_i,\ t_{i+1})$, one gets

$$\mathcal{L}V_{p_{i+1}}\left(\xi_{av}(t),\xi_{a'v}(t)\right)\leq-\kappa V_{p_{i+1}}\left(\xi_{av}(t),\xi_{a'v}(t)\right).$$

Similar to the proof of Theorem 2, for all $i\in\mathbb{N}_0$ and $t\in[t_i,\ t_{i+1}]$, we have

$$\mathbb{E}\left[V_{p_{i+1}}(\xi_{av}(t),\xi_{a'v}(t))\right]\leq\mathbb{E}\left[V_{p_{i+1}}(\xi_{av}(t_i),\xi_{a'v}(t_i))\right]\mathsf{e}^{-\kappa(t-t_i)}.\qquad(20)$$

Particularly, for $t=t_{i+1}$ and from (5), it can be checked that for all $i\in\mathbb{N}_0$:

$$\mathbb{E}\left[V_{p_{i+2}}(\xi_{av}(t_{i+1}),\xi_{a'v}(t_{i+1}))\right]\leq\mu\mathsf{e}^{-\kappa(t_{i+1}-t_i)}\mathbb{E}\left[V_{p_{i+1}}(\xi_{av}(t_i),\xi_{a'v}(t_i))\right].$$

Using this inequality, we prove by induction that for all $i\in\mathbb{N}_0$

$$\mathbb{E}\left[V_{p_{i+1}}(\xi_{av}(t_i),\xi_{a'v}(t_i))\right]\leq\mu^i\mathsf{e}^{-\kappa t_i}\mathbb{E}\left[V_{p_1}(a,a')\right].\qquad(21)$$

From (20) and (21), for all $i\in\mathbb{N}_0$ and $t\in[t_i,\ t_{i+1}]$, one obtains

$$\mathbb{E}\left[V_{p_{i+1}}(\xi_{av}(t),\xi_{a'v}(t))\right]\leq\mu^i\mathsf{e}^{-\kappa t}\mathbb{E}\left[V_{p_1}(a,a')\right].$$

Since the switching signal v has dwell time τ_d, then $t_i\geq i\tau_d$ and hence for all $t\in[t_i,\ t_{i+1}]$, $t\geq i\tau_d$. Since $\mu\geq1$, then for all $i\in\mathbb{N}_0$ and $t\in[t_i,\ t_{i+1}]$, one has

$$\mu^i=\mathsf{e}^{i\log\mu}\leq\mathsf{e}^{(\log\mu/\tau_d)t}.$$

Therefore, for all $i\in\mathbb{N}_0$ and $t\in[t_i,\ t_{i+1}]$, we get

$$\mathbb{E}\left[V_{p_{i+1}}(\xi_{av}(t),\xi_{a'v}(t))\right]\leq\mathsf{e}^{((\log\mu/\tau_d)-\kappa)t}\mathbb{E}\left[V_{p_1}(a,a')\right].$$

Using functions $\underline{\alpha},\overline{\alpha}$ and Jensen's inequality, and for all $t\in\mathbb{R}_0^+$, where $t\in[t_i,t_{i+1}]$ for some $i\in\mathbb{N}_0$, we have

$$\begin{aligned}&\underline{\alpha}\left(\mathbb{E}\left[\|\xi_{av}(t)-\xi_{a'v}(t)\|^q\right]\right)\leq\underline{\alpha}_{p_{i+1}}\left(\mathbb{E}\left[\|\xi_{av}(t)-\xi_{a'v}(t)\|^q\right]\right)\leq\mathbb{E}\left[\underline{\alpha}_{p_{i+1}}\left(\|\xi_{av}(t)-\xi_{a'v}(t)\|^q\right)\right]\\&\leq\mathbb{E}\left[V_{p_{i+1}}(\xi_{av}(t),\xi_{a'v}(t))\right]\leq\mathsf{e}^{((\log\mu/\tau_d)-\kappa)t}\mathbb{E}\left[V_{p_1}(a,a')\right]\leq\mathsf{e}^{((\log\mu/\tau_d)-\kappa)t}\mathbb{E}\left[\overline{\alpha}_{p_1}\left(\|a-a'\|^q\right)\right]\\&\leq\mathsf{e}^{((\log\mu/\tau_d)-\kappa)t}\overline{\alpha}_{p_1}\left(\mathbb{E}\left[\|a-a'\|^q\right]\right)\leq\mathsf{e}^{((\log\mu/\tau_d)-\kappa)t}\overline{\alpha}\left(\mathbb{E}\left[\|a-a'\|^q\right]\right).\end{aligned}$$

Therefore, for all $t \in \mathbb{R}_0^+$

$$\mathbb{E}\left[\|\xi_{av}(t) - \xi_{a'v}(t)\|^q\right] \leq \underline{\alpha}^{-1}\left(\mathsf{e}^{((\log \mu/\tau_d) - \kappa)t}\overline{\alpha}\left(\mathbb{E}\left[\|a - a'\|^q\right]\right)\right).$$

Then condition (4) holds with the function $\beta(r, s) = \underline{\alpha}^{-1}\left(\overline{\alpha}(r)\mathsf{e}^{((\log \mu/\tau_d) - \kappa)s}\right)$ which is a $\mathcal{KL}$ function because by assumption $\log \mu/\tau_d - \kappa < 0$. The same inequality can be shown for switching signals with a finite number of discontinuities. Therefore, the stochastic switched system Σ_{τ_d} is δ-GUAS-M$_q$. □

Proof (of Lemma 1). It is not difficult to check that the function V_p in (6) satisfies properties (i) and (ii) of Definition 4 with functions $\underline{\alpha}_p(y) := \frac{1}{q}\left(\lambda_{\min}\left(P_p\right)\right)^{q/2} y$ and $\overline{\alpha}_p(y) := \frac{1}{q}\left(n\lambda_{\max}\left(P_p\right)\right)^{q/2} y$. It then suffices to verify property (iii). We verify property (iii) for the case that f_p is differentiable and using condition (8). The proof, using condition (7), is completely similar by just removing the inequality in the proof including derivative of f_p. By the definition of V_p in (6), for any $x, x' \in \mathbb{R}^n$ such that $x \neq x'$, and for $q \in \{1, 2\}$, one has

$$\begin{aligned}
\partial_x V_p &= -\partial_{x'} V_p = (x - x')^T P_p\left(\widetilde{V}(x, x')\right)^{q/2-1}, \\
\partial_{x,x} V_p &= \partial_{x',x'} V_p = -\partial_{x,x'} V_p \\
&= P_p\left(\widetilde{V}(x, x')\right)^{q/2-1} + \frac{q-2}{q} P_p(x - x')(x - x')^T P_p\left(\widetilde{V}(x, x')\right)^{q/2-2}.
\end{aligned}$$

Therefore, following the definition of $\mathcal{L}$, and for any $x, x', z \in \mathbb{R}^n$ such that $x \neq x'$, one obtains:

$$\begin{aligned}
&\mathcal{L}V_p(x, x') \\
&= (x - x')^T P_p\left(\widetilde{V}(x, x')\right)^{q/2-1}\left(f_p(x) - f_p(x')\right) \\
&\quad + \frac{1}{2}\mathrm{Tr}\left(\begin{bmatrix} g_p(x) \\ g_p(x') \end{bmatrix}\begin{bmatrix} g_p^T(x) & g_p^T(x') \end{bmatrix}\begin{bmatrix} \partial_{x,x} V_p & -\partial_{x,x} V_p \\ -\partial_{x,x} V_p & \partial_{x,x} V_p \end{bmatrix}\right) \\
&= (x - x')^T P_p\left(\widetilde{V}(x, x')\right)^{q/2-1}\left(f_p(x) - f_p(x')\right) \\
&\quad + \frac{1}{2}\mathrm{Tr}\left(\left(g_p(x) - g_p(x')\right)\left(g_p^T(x) - g_p^T(x')\right)\partial_{x,x} V_p\right) \\
&= (x - x')^T P_p\left(\widetilde{V}(x, x')\right)^{q/2-1}\left(f_p(x) - f_p(x')\right) \\
&\quad + \frac{1}{2}\left\|\sqrt{P_p}\left(g_p(x) - g_p(x')\right)\right\|_F^2\left(\widetilde{V}(x, x')\right)^{q/2-1} \\
&\quad + \frac{q-2}{q}\left\|(x - x')^T P_p\left(g_p(x) - g_p(x')\right)\right\|_F^2\left(\widetilde{V}(x, x')\right)^{q/2-2} \\
&\leq (x - x')^T P_p\left(\widetilde{V}(x, x')\right)^{q/2-1}\left(f_p(x) - f_p(x')\right) \\
&\quad + \frac{1}{2}\left\|\sqrt{P_p}\left(g_p(x) - g_p(x')\right)\right\|_F^2\left(\widetilde{V}(x, x')\right)^{q/2-1} \\
&\leq \left((x - x')^T P_p \partial_x f_p(z)(x - x') + \frac{1}{2}\left\|\sqrt{P_p}\left(g_p(x) - g_p(x')\right)\right\|_F^2\right)\left(\widetilde{V}(x, x')\right)^{q/2-1} \qquad (22) \\
&\leq -\kappa_p V_p(x, x').
\end{aligned}$$

In (22), $z \in \mathbb{R}^n$ where the mean value theorem is applied to the differentiable function $x \mapsto f_p(x)$ at points x, x'. $\square$

Proof (of Corollary 1). The corollary is a particular case of Lemma 1. It suffices to show that for affine dynamics the LMI (9) yields the condition in (8). First it is straightforward to observe that

$$\begin{aligned}\left\|\sqrt{P_p}\left(g_p(x)-g_p(x')\right)\right\|_F^2 &= \mathrm{Tr}\left(\left(g_p(x)-g_p(x')\right)^T P_p \left(g_p(x)-g_p(x')\right)\right)\\ &= \left(x-x'\right)^T \sum_{i=1}^{q} \sigma_{i_p}^T P_p \sigma_{i_p}(x-x'),\end{aligned}$$

and that

$$(x-x')^T P_p \partial_x f_p(z)(x-x') = \frac{1}{2}(x-x')^T \left(P_p A_p + A_p^T P_p\right)(x-x'),$$

for any $x, x', z \in \mathbb{R}^n$. Now suppose there exists $\widehat{\kappa}_p \in \mathbb{R}^+$ such that (9) holds. It can be readily verified that the desired assertion of (8) is verified by choosing $\kappa_p = \widehat{\kappa}_p/2$. $\square$

Proof (of Theorem 4). We start by proving $S_\tau(\Sigma) \preceq_{\mathcal{S}}^{\varepsilon} S_{\mathsf{q}}(\Sigma)$. Consider the relation $R \subseteq X_\tau \times X_{\mathsf{q}}$ defined by $(x_\tau, x_{\mathsf{q}}) \in R$ if and only if $\mathbb{E}\left[V\left(H_\tau(x_\tau), H_{\mathsf{q}}(x_{\mathsf{q}})\right)\right] = \mathbb{E}\left[V\left(x_\tau, x_{\mathsf{q}}\right)\right] \leq \underline{\alpha}(\varepsilon)$. Since $X_{\tau 0} \subseteq \bigcup_{p\in[\mathbb{R}^n]_\eta} \mathcal{B}_\eta(p)$, for every $x_{\tau 0} \in X_{\tau 0}$ there always exists $x_{\mathsf{q}0} \in X_{\mathsf{q}0}$ such that

$$\mathbb{E}\left[\|x_{\tau 0} - x_{\mathsf{q}0}\|\right] = \|x_{\tau 0} - x_{\mathsf{q}0}\| \leq \eta.$$

Then,

$$\mathbb{E}\left[V(x_{\tau 0}, x_{\mathsf{q}0})\right] = V(x_{\tau 0}, x_{\mathsf{q}0}) \leq \overline{\alpha}\left(\|x_{\tau 0} - x_{\mathsf{q}0}\|^q\right) \leq \overline{\alpha}\left(\eta^q\right) \leq \underline{\alpha}(\varepsilon),$$

because of (12) and since $\overline{\alpha}$ is a $\mathcal{K}_\infty$ function. Hence, $(x_{\tau 0}, x_{\mathsf{q}0}) \in R$ and condition (i) in Definition 6 is satisfied. Now consider any $(x_\tau, x_{\mathsf{q}}) \in R$. Condition (ii) in Definition 6 is satisfied because

$$\mathbb{E}\left[\|x_\tau - x_{\mathsf{q}}\|^q\right] \leq \underline{\alpha}^{-1}\left(\mathbb{E}\left[V(x_\tau, x_{\mathsf{q}})\right]\right) \leq \varepsilon. \tag{23}$$

We used the convexity assumption of $\underline{\alpha}$ and the Jensen inequality [24] to show the inequalities in (23). Let us now show that condition (iii) in Definition 6 holds. Consider the transition $x_\tau \xrightarrow[\tau]{p} x'_\tau = \xi_{x_\tau p}(\tau)$ in $S_\tau(\Sigma)$. Since V is a common Lyapunov function for Σ, we have

$$\mathbb{E}\left[V(x'_\tau, \xi_{x_{\mathsf{q}}p}(\tau))\right] \leq \mathbb{E}\left[V(x_\tau, x_q)\right] \mathsf{e}^{-\kappa\tau} \leq \underline{\alpha}(\varepsilon)\mathsf{e}^{-\kappa\tau}. \tag{24}$$

Since $\mathbb{R}^n \subseteq \bigcup_{p\in[\mathbb{R}^n]_\eta} \mathcal{B}_\eta(p)$, there exists $x'_{\mathsf{q}} \in X_{\mathsf{q}}$ such that

$$\left\|\overline{\xi}_{x_{\mathsf{q}}p}(\tau) - x'_{\mathsf{q}}\right\| \leq \eta, \tag{25}$$

which, by the definition of $S_{\mathsf{q}}(\Sigma)$, implies the existence of $x_{\mathsf{q}} \xrightarrow[\mathsf{q}]{p} x'_{\mathsf{q}}$ in $S_{\mathsf{q}}(\Sigma)$. Using Lemma 2, the concavity of $\widehat{\gamma}$, the Jensen inequality [24], the inequalities (10), (13), (24), (25), and triangle inequality, we obtain

$$\begin{aligned}\mathbb{E}\left[V(x'_\tau, x'_{\mathsf{q}})\right] &= \mathbb{E}\left[V(x'_\tau, \xi_{x_{\mathsf{q}}p}(\tau)) + V(x'_\tau, x'_{\mathsf{q}}) - V(x'_\tau, \xi_{x_{\mathsf{q}}p}(\tau))\right] \\ &= \mathbb{E}\left[V(x'_\tau, \xi_{x_{\mathsf{q}}p}(\tau))\right] + \mathbb{E}\left[V(x'_\tau, x'_{\mathsf{q}}) - V(x'_\tau, \xi_{x_{\mathsf{q}}p}(\tau))\right] \\ &\le \underline{\alpha}(\varepsilon)\mathsf{e}^{-\kappa\tau} + \mathbb{E}\left[\widehat{\gamma}\left(\left\|\xi_{x_{\mathsf{q}}p}(\tau) - x'_{\mathsf{q}}\right\|\right)\right] \\ &\le \underline{\alpha}(\varepsilon)\mathsf{e}^{-\kappa\tau} + \widehat{\gamma}\left(\mathbb{E}\left[\left\|\xi_{x_{\mathsf{q}}p}(\tau) - \overline{\xi}_{x_{\mathsf{q}}p}(\tau) + \overline{\xi}_{x_{\mathsf{q}}p}(\tau) - x'_{\mathsf{q}}\right\|\right]\right) \\ &\le \underline{\alpha}(\varepsilon)\mathsf{e}^{-\kappa\tau} + \widehat{\gamma}\left(\mathbb{E}\left[\left\|\xi_{x_{\mathsf{q}}p}(\tau) - \overline{\xi}_{x_{\mathsf{q}}p}(\tau)\right\|\right] + \left\|\overline{\xi}_{x_{\mathsf{q}}p}(\tau) - x'_{\mathsf{q}}\right\|\right) \\ &\le \underline{\alpha}(\varepsilon)\mathsf{e}^{-\kappa\tau} + \widehat{\gamma}\left(h(\tau, G)\mathsf{e}^{-\kappa\tau} + \eta\right) \le \underline{\alpha}(\varepsilon).\end{aligned}$$

Therefore, we conclude that $\left(x'_\tau, x'_{\mathsf{q}}\right) \in R$ and that condition (iii) in Definition 6 holds. In a similar way, we can prove that $S_{\mathsf{q}}(\Sigma) \preceq^{\varepsilon}_{\mathcal{S}} S_\tau(\Sigma)$ implying that R is an ε-approximate bisimulation relation between $S_{\mathsf{q}}(\Sigma)$ and $S_\tau(\Sigma)$. □

Proof (of Theorem 5). The proof was inspired by the proof of Theorem 4.2 in [15] for non-probabilistic switched systems. We start by proving $S_\tau\left(\Sigma_{\tau_d}\right) \preceq^{\varepsilon}_{\mathcal{S}} S_{\mathsf{q}}\left(\Sigma_{\tau_d}\right)$. Consider the relation $R \subseteq X_\tau \times X_{\mathsf{q}}$ defined by $(x_\tau, p_1, i_1, x_{\mathsf{q}}, p_2, i_2) \in R$ if and only if $p_1 = p_2 = p$, $i_1 = i_2 = i$, and $\mathbb{E}\left[V_p\left(H_\tau(x_\tau, p_1, i_1), H_{\mathsf{q}}(x_{\mathsf{q}}, p_2, i_2)\right)\right] = \mathbb{E}\left[V_p\left(x_\tau, x_{\mathsf{q}}\right)\right] \le \delta_i$, where $\delta_0, \cdots, \delta_N$ are given recursively by

$$\delta_0 = \underline{\alpha}(\varepsilon), \;\; \delta_{i+1} = \mathsf{e}^{-\kappa\tau}\delta_i + \widehat{\gamma}\left(h(\tau, G)\mathsf{e}^{-\kappa\tau} + \eta\right).$$

One can easily verify that

$$\begin{aligned}\delta_i =& \mathsf{e}^{-i\kappa\tau}\underline{\alpha}(\varepsilon) + \widehat{\gamma}\left(h(\tau, G)\mathsf{e}^{-\kappa\tau} + \eta\right)\frac{1 - \mathsf{e}^{-i\kappa\tau}}{1 - \mathsf{e}^{-\kappa\tau}} \\ =& \frac{\widehat{\gamma}\left(h(\tau, G)\mathsf{e}^{-\kappa\tau} + \eta\right)}{1 - \mathsf{e}^{-\kappa\tau}} + \mathsf{e}^{-i\kappa\tau}\left(\underline{\alpha}(\varepsilon) - \frac{\widehat{\gamma}\left(h(\tau, G)\mathsf{e}^{-\kappa\tau} + \eta\right)}{1 - \mathsf{e}^{-\kappa\tau}}\right).\end{aligned} \tag{26}$$

Since $\mu \ge 1$, and from (17), one has

$$\widehat{\gamma}\left(h(\tau, G)\mathsf{e}^{-\kappa\tau} + \eta\right) \le (1 - \mathsf{e}^{-\kappa\tau})\underline{\alpha}(\varepsilon).$$

It follows from (26) that $\delta_0 \ge \delta_1 \ge \cdots \ge \delta_{N-1} \ge \delta_N$. From (17), and since $\tau_d = N\tau$, we get

$$\begin{aligned}\delta_N =& \mathsf{e}^{-\kappa\tau_d}\underline{\alpha}(\varepsilon) + \widehat{\gamma}\left(h(\tau, G)\mathsf{e}^{-\kappa\tau} + \eta\right)\frac{1 - \mathsf{e}^{-\kappa\tau_d}}{1 - \mathsf{e}^{-\kappa\tau}} \\ \le& \mathsf{e}^{-\kappa\tau_d}\underline{\alpha}(\varepsilon) + \left(\frac{1}{\mu} - \mathsf{e}^{-\kappa\tau_d}\right)\underline{\alpha}(\varepsilon) = \frac{\underline{\alpha}(\varepsilon)}{\mu}.\end{aligned} \tag{27}$$

We can now prove that R is an ε-approximate simulation relation from $S_\tau\left(\Sigma_{\tau_d}\right)$ to $S_{\mathsf{q}}\left(\Sigma_{\tau_d}\right)$. Since $\mathcal{X}_0 \subseteq \bigcup_{p\in[\mathbb{R}^n]_\eta} \mathcal{B}_\eta(p)$, for every $(x_{\tau 0}, p, 0) \in X_{\tau 0}$ there always exists $(x_{\mathsf{q}0}, p, 0) \in X_{\mathsf{q}0}$ such that $\|x_{\tau 0} - x_{\mathsf{q}0}\| \le \eta$. Then,

$$\begin{aligned}\mathbb{E}\left[V_p(H_\tau(x_{\tau 0}, p, 0), H_{\mathsf{q}}(x_{\mathsf{q}0}, p, 0)\right] = V_p(x_{\tau 0}, x_{\mathsf{q}0}) &\le \overline{\alpha}_p\left(\|x_{\tau 0} - x_{\mathsf{q}0}\|^q\right) \\ &\le \overline{\alpha}\left(\|x_{\tau 0} - x_{\mathsf{q}0}\|^q\right) \le \overline{\alpha}\left(\eta^q\right) \le \underline{\alpha}(\varepsilon),\end{aligned}$$

because of (16) and since $\overline{\alpha}$ is a $\mathcal{K}_\infty$ function. Hence, $V_p(x_{\tau 0}, x_{\mathsf{q}0}) \le \delta_0$ and $(x_{\tau 0}, p, 0, x_{\mathsf{q}0}, p, 0) \in R$ and condition (i) in Definition 6 is satisfied. Now consider any $(x_\tau, p, i, x_{\mathsf{q}}, p, i) \in R$. Using the convexity assumption of $\underline{\alpha}_p$, and since it is a $\mathcal{K}_\infty$ function, and the Jensen inequality [24], We have:

$$\underline{\alpha}\left(\mathbb{E}\left[\|H_\tau(x_\tau, p, i) - H_{\mathsf{q}}(x_{\mathsf{q}}, p, i)\|^q\right]\right) = \underline{\alpha}\left(\mathbb{E}\left[\|x_\tau - x_{\mathsf{q}}\|^q\right]\right) \leq \underline{\alpha}_p\left(\mathbb{E}\left[\|x_\tau - x_{\mathsf{q}}\|^q\right]\right)$$
$$\leq \mathbb{E}\left[\underline{\alpha}_p\left(\|x_\tau - x_{\mathsf{q}}\|^q\right)\right] \leq \mathbb{E}\left[V_p(x_\tau, x_{\mathsf{q}})\right] \leq \delta_i \leq \delta_0.$$

Therefore, we obtain

$$\mathbb{E}\left[\|x_\tau - x_{\mathsf{q}}\|^q\right] \leq \underline{\alpha}^{-1}\left(\delta_0\right) \leq \varepsilon, \tag{28}$$

because of $\underline{\alpha} \in \mathcal{K}_\infty$. Hence, condition (ii) in Definition 6 is satisfied. Let us now show that condition (iii) in Definition 6 holds. Consider the transition $(x_\tau, p, i) \xrightarrow[\tau]{p} (x'_\tau, p', i')$ in $S_\tau\left(\Sigma_{\tau_d}\right)$, where $x'_\tau = \xi_{x_\tau p}(\tau)$. Since V_p is a δ-GAS-M$_q$ Lyapunov function for subsystem Σ_p, we have

$$\mathbb{E}\left[V_p(x'_\tau, \xi_{x_{\mathsf{q}}p}(\tau))\right] \leq \mathbb{E}\left[V_p(x_\tau, x_q)\right] \mathsf{e}^{-\kappa\tau} \leq \mathsf{e}^{-\kappa\tau}\delta_i. \tag{29}$$

Since $\mathbb{R}^n \subseteq \bigcup_{p \in [\mathbb{R}^n]_\eta} \mathcal{B}_\eta(p)$, there exists $x'_{\mathsf{q}} \in [\mathbb{R}^n]_\eta$ such that

$$\left\|\overline{\xi}_{x_{\mathsf{q}}p}(\tau) - x'_{\mathsf{q}}\right\| \leq \eta. \tag{30}$$

Using Lemma 2, the $\mathcal{K}_\infty$ function $\widehat{\gamma}$, the concavity of $\widehat{\gamma}_p$, the Jensen inequality [24], the inequalities (10), (29), (30), and triangle inequality, we obtain

$$\begin{aligned}
\mathbb{E}\left[V_p(x'_\tau, x'_{\mathsf{q}})\right] &= \mathbb{E}\left[V_p(x'_\tau, \xi_{x_{\mathsf{q}}p}(\tau)) + V_p(x'_\tau, x'_{\mathsf{q}}) - V_p(x'_\tau, \xi_{x_{\mathsf{q}}p}(\tau))\right] \\
&= \mathbb{E}\left[V_p(x'_\tau, \xi_{x_{\mathsf{q}}p}(\tau))\right] + \mathbb{E}\left[V_p(x'_\tau, x'_{\mathsf{q}}) - V_p(x'_\tau, \xi_{x_{\mathsf{q}}p}(\tau))\right] \\
&\leq \mathsf{e}^{-\kappa\tau}\delta_i + \mathbb{E}\left[\widehat{\gamma}_p\left(\left\|\xi_{x_{\mathsf{q}}p}(\tau) - x'_{\mathsf{q}}\right\|\right)\right] \\
&\leq \mathsf{e}^{-\kappa\tau}\delta_i + \widehat{\gamma}_p\left(\mathbb{E}\left[\left\|\xi_{x_{\mathsf{q}}p}(\tau) - x'_{\mathsf{q}}\right\|\right]\right) \\
&\leq \mathsf{e}^{-\kappa\tau}\delta_i + \widehat{\gamma}\left(\mathbb{E}\left[\left\|\xi_{x_{\mathsf{q}}p}(\tau) - \overline{\xi}_{x_{\mathsf{q}}p}(\tau) + \overline{\xi}_{x_{\mathsf{q}}p}(\tau) - x'_{\mathsf{q}}\right\|\right]\right) \\
&\leq \mathsf{e}^{-\kappa\tau}\delta_i + \widehat{\gamma}\left(\mathbb{E}\left[\left\|\xi_{x_{\mathsf{q}}p}(\tau) - \overline{\xi}_{x_{\mathsf{q}}p}(\tau)\right\|\right] + \left\|\overline{\xi}_{x_{\mathsf{q}}p}(\tau) - x'_{\mathsf{q}}\right\|\right) \\
&\leq \mathsf{e}^{-\kappa\tau}\delta_i + \widehat{\gamma}\left(h(\tau, G)\mathsf{e}^{-\kappa\tau} + \eta\right) = \delta_{i+1}.
\end{aligned} \tag{31}$$

We now examine three separate cases:

- If $i < N-1$, then $p' = p$, and $i' = i+1$; since, from (31), $\mathbb{E}\left[V_p(x'_\tau, x'_{\mathsf{q}})\right] \leq \delta_{i+1}$, we conclude that $(x'_\tau, p, i+1, x'_{\mathsf{q}}, p, i+1) \in R$;
- If $i = N-1$, and $p' = p$, then $i' = N-1$; from (31), $\mathbb{E}\left[V_p(x'_\tau, x'_{\mathsf{q}})\right] \leq \delta_N \leq \delta_{N-1}$, we conclude that $(x'_\tau, p, N-1, x'_{\mathsf{q}}, p, N-1) \in R$;
- If $i = N-1$, and $p' \neq p$, then $i' = 0$; from (27) and (31), $\mathbb{E}\left[V_p(x'_\tau, x'_{\mathsf{q}})\right] \leq \delta_N \leq \delta_0/\mu$. From (5), it follows that $\mathbb{E}\left[V_{p'}(x'_\tau, x'_{\mathsf{q}})\right] \leq \mu\mathbb{E}\left[V_p(x'_\tau, x'_{\mathsf{q}})\right] \leq \delta_0$. Hence, $(x'_\tau, p', 0, x'_{\mathsf{q}}, p', 0) \in R$.

Therefore, we conclude that condition (iii) in Definition 6 holds. In a similar way, we can prove that $S_{\mathsf{q}}\left(\Sigma_{\tau_d}\right) \preceq^{\varepsilon}_{\mathcal{S}} S_\tau\left(\Sigma_{\tau_d}\right)$ implying that R is an ε-approximate bisimulation relation between $S_{\mathsf{q}}(\Sigma_{\tau_d})$ and $S_\tau(\Sigma_{\tau_d})$. □

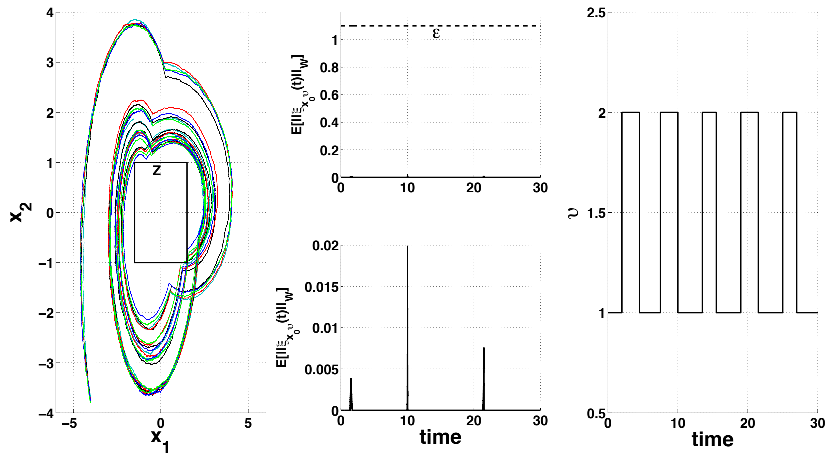

Fig. 3: Several realizations of the closed-loop trajectory $\xi_{x_0\upsilon}$ with initial condition $x_0 = (-4, -3.8)$ (left panel). Average values (over 100 experiments) in time of the distance between solution process $\xi_{x_0\upsilon}$ and set $W = D\backslash Z$, in different vertical scales (middle panel). Evolution of the synthesized switching signal υ (right panel).

Case study with multiple Lyapunov functions: Consider the following stochastic switched system borrowed from [15] and additionally affected by noise:

$$\Sigma : \left\{ \begin{bmatrix} \mathrm{d}\,\xi_1 \\ \mathrm{d}\,\xi_2 \end{bmatrix} = \left(\begin{bmatrix} -0.25 & p \\ p-3 & -0.25 \end{bmatrix} + (-1)^p \begin{bmatrix} 0.25 \\ 3-p \end{bmatrix} \right) \mathrm{d}\,t + \begin{bmatrix} 0.02\xi_1 \,\mathrm{d}\,W_t^1 \\ 0.02\xi_2 \,\mathrm{d}\,W_t^2 \end{bmatrix}, \right. \quad (32)$$

where $p = 1, 2$. A noise-free version of the Σ is endowed with stable subsystems, however it can globally exhibit unstable behaviors for some switching signals [15]. Similarly, Σ does not admit a common δ-GAS-M$_q$ Lyapunov function. We're left with the option of seeking for multiple Lyapunov functions. It can be indeed shown that each subsystem Σ_p has a δ-GAS-M$_1$ Lyapunov function of the form $V_p(x_1, x_2) = \sqrt{(x_1 - x_2)^T P_p (x_1 - x_2)}$, with

$$P_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \; P_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

By definition, these δ-GAS-M$_1$ Lyapunov functions have the following characteristics: $\underline{\alpha}(r) = r$, $\overline{\alpha}(r) = 2r$, $\kappa = 0.2498$. Furthermore, the assumptions of Theorem 3 hold by choosing a parameter $\mu = \sqrt{2}$ and a dwell time $\tau_d = 2 > \log \mu / \kappa$. In conclusion, the stochastic switched system Σ is δ-GUAS-M$_1$.

Let us work within the set $D = [-6,\ 6] \times [-4,\ 4]$ of the state space of Σ. For a sampling time $\tau = 0.5$, using the inequality (18) the precision ε is lower bounded by 1.07. For a precision $\varepsilon = 1.2$, the discretization parameter η for $S_{\mathsf{q}}(\Sigma)$, obtained from Theorem 5, is equal to 0.024. The resulting number of states in $S_{\mathsf{q}}(\Sigma)$ is 167835. The CPU time needed for computing the abstraction has amounted to 94.03 seconds.

Now, consider the objective to design a controller (switching policy) forcing the first moment of the trajectories of Σ to stay within D while always avoiding the set $Z = [-1.5,\ 1.5] \times [-1,\ 1]$. This corresponds to the following LTL specification: $\Box\,(D\backslash Z)$.

The CPU time needed for synthesizing the controller has amounted to 11.81 seconds. Figure 3 displays several realizations of the closed-loop trajectory of $\xi_{x_0\upsilon}$, stemming from the deterministic initial condition $x_0 = (-4, -3.8)$ (left panel), as well as the corresponding evolution of the switching signal υ (right panel). Furthermore, Figure 3 (middle panel) shows the average value (over 100 experiments) of the distance in time between the solution process $\xi_{x_0\upsilon}$ and the set $D\backslash Z$, namely $\|\xi_{x_0\upsilon}(t)\|_{D\backslash Z}$. Notice that the empirical average distance is significantly lower than the theoretical precision $\varepsilon = 1.2$ (as discussed, theoretical bounds can be improved by seeking optimized Lyapunov functions).