Linear quadratic regulation of polytopic time-inhomogeneous Markov jump linear systems (extended version)

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Abstract— In most real cases transition probabilities between operational modes of Markov jump linear systems cannot be computed exactly and are time-varying. We take into account this aspect by considering Markov jump linear systems where the underlying Markov chain is polytopic and time-inhomogeneous, i.e. its transition probability matrix is varying over time, with variations that are arbitrary within a polytopic set of stochastic matrices. We address and solve for this class of systems the infinite-horizon optimal control problem. In particular, we show that the optimal controller can be obtained from a set of coupled algebraic Riccati equations, and that for mean square stabilizable systems the optimal finite-horizon cost corresponding to the solution to a parsimonious set of coupled difference Riccati equations converges exponentially fast to the optimal infinite-horizon cost related to the set of coupled algebraic Riccati equations. All the presented concepts are illustrated on a numerical example showing the efficiency of the provided solution.

I. INTRODUCTION

In discrete-time Markov jump linear systems (MJLSs), the transition probabilities of jumps between operational modes are fundamental in determining the dynamic behaviour [1]. These transition probabilities are generally considered to be either time-invariant or certain in the majority of dedicated studies, see [2] as a textbook presenting important results and detailed examination of the general state of the art.

In most real cases, however, the transition probability matrices (TPMs) are affected by global uncertainty due to random and systematic errors of measurement and numerical computation procedures (used to obtain the values of TPMs), by incomplete knowledge of some transition probabilities (when adequate samples of the transitions are costly or time-consuming to obtain), and by abrupt and unpredictable time-variance (due to environmental factors, like for instance the wind perturbing the model of airspeed variation in a vertical take-off landing helicopter system [3]). The study of robustness to such variations is naturally important in many applications, especially for wireless networked control systems (see e.g. [4], [5], [6], [7] and references therein for a general overview)

In this work, we allow for incomplete knowledge and time-varying uncertainties in transition probabilities by studying polytopic time-inhomogeneous (PTI) MJLSs, where the

TPMs are unknown and time-varying within a bounded set. This model permits to include realistic and reasonable uncertainties while still maintaining the problem convex. In fact, there exists a considerable number of works on discrete-time Markov jump systems (both linear and nonlinear) with polytopic uncertainties, which can be either time-varying or time-invariant, as extensively discussed in [8, Section 1.5].

We build upon our previous results on stability [9] and optimal finite-horizon control [10] of PTI MJLSs, providing an analytical solution for the infinite-horizon optimal state-feedback control problem, that can be computed efficiently. Specifically, we show that for a stabilizable system, the optimal steady-state controller is obtained from a set of coupled algebraic Riccati equations (CAREs). The cardinality of this set of CAREs equals to the number of vertices of the convex polytope characterizing the time-varying uncertainties in TPMs. Furthermore, when a PTI MJLS is stabilizable (in the mean square sense), the optimal finite-horizon cost of the robust control obtained from the solution to a parsimonious set of coupled difference Riccati equations (CDREs) converges exponentially fast to the optimal infinite-horizon cost related to the set of CAREs. From a technical point of view these results are a nontrivial extension of [1], [9] and [10], since they require a proper definition of an appropriate set of CAREs, which solution exists, is unique, and achieves the optimal quadratic cost, while the convergence between the finite- and infinite-horizon controllers is ensured in terms of costs of control actions and not in terms of the Riccati equations themselves. The obtained results are validated on a numerical example based on the Samuelson's multiplier-accelerator model with three operational modes, three to four polytopic bounds on a time-inhomogeneous TPM and different time horizons, showing that for a stabilizable and detectable system both finite- and infinite-horizon control problems can be solved efficiently by taking advantage of parsimonious sets.

This paper is organized as follows. After presenting the PTI model of MJLSs, in Section II, we summarize our result on finite-horizon optimal robust state-feedback control in a way useful to formulate and solve the problem for the infinite-time horizon. In Section III, we recall the concept of the stability equivalence in PTI setting and formally define the notion of mean square stabilizability and mean square detectability. In Section IV, we define the stabilizing solution to the control CAREs, show that, for mean square stabilizable and detectable PTI MJLSs, this solution is unique, it achieves the optimal infinite-horizon cost of robust state-feedback control, and the optimal finite-horizon cost converges to it. In

Section V, we showcase our results on a numerical example. Lastly, Section VI concludes the paper.

A. Notation

We denote the set of all either real or complex numbers by $\mathbb{F}$, and the sets of integers, of all nonnegative integers, and of all positive integers by $\mathbb{Z}$, $\mathbb{Z}_0$, and $\mathbb{Z}_+$, respectively. An n-dimensional linear space with entries in $\mathbb{F}$ is indicated by $\mathbb{F}^n$, while a set of matrices with m rows, n columns, and entries in $\mathbb{F}$ is denoted by $\mathbb{F}^{m,n}$. The sets of positive definite and positive semi-definite matrices of order n are indicated by $\mathbb{F}_+^{n,n}$ and $\mathbb{F}_0^{n,n}$, respectively. As alternative notation to $A \in \mathbb{F}_0^{n,n}$ (respectively to $A \in \mathbb{F}_+^{n,n}$) we also write $A \succeq 0$ (respectively $A \succ 0$). The identity matrix of size n is denoted by I_n. The operation of transposition is indicated by superscript ^{T}, the complex conjugation by overbar, while the conjugate transpose of a (complex) matrix is denoted by superscript *. Clearly, for a set of real matrices, the transpose and conjugate transpose are the same. Then, $\otimes$ denotes the Kronecker product, and $\oplus$ stands for the direct sum. Notably, the direct sum of a sequence of square matrices $\boldsymbol{A} = (A_i)_{i=1}^N$ produces a block diagonal matrix, having the elements of $\boldsymbol{A}$ on the main diagonal blocks. The spectral radius of a square matrix is indicated by $\rho(\cdot)$, while the joint spectral radius of a set of square matrices is denoted by $\hat{\rho}(\cdot)$. Finally, $\|\cdot\|$ stands for (induced) Euclidean norm, conv indicates the convex hull of a nonempty set, $\mathbb{E}(\cdot)$ denotes the expected value.

II. PROBLEM STATEMENT

Consider the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_k), \mathrm{Pr})$, where Ω is the sample space, $\mathcal{F}$ is the σ-algebra of (Borel) measurable events, $(\mathcal{F}_k)$ is the related filtration, and Pr is the probability measure. An MJLS defined on this stochastic basis is represented by the following dynamical system

$$\begin{cases} \mathrm{x}_{k+1} = A_{\theta_k}\mathrm{x}_k + B_{\theta_k}\mathrm{u}_k, \\ \mathrm{z}_k = C_{\theta_k}\mathrm{x}_k + D_{\theta_k}\mathrm{u}_k, \end{cases} \quad (1)$$

where $\mathrm{x}_k \in \mathbb{F}^{n_x}$ is the state vector, $k \in \mathbb{Z}_0$ is a discrete-time instant, $\mathrm{u}_k \in \mathbb{F}^{n_u}$ is the controlled input, and $\mathrm{z}_k \in \mathbb{F}^{n_z}$ is the vector of measured system output variables. Then, $\theta: \mathbb{Z}_0 \times \Omega \to \mathbb{M}$ is a discrete-time Markov chain, that takes values in a finite set of operational modes $\mathbb{M} \triangleq \{i\}_{i=1}^N$, so for every operational mode there is a correspondent system matrix of appropriate size, and the collection of the system matrices of each type is represented by a sequence of N matrices. Specifically, $\boldsymbol{A} \triangleq (A_i)_{i=1}^N$ is the sequence of state matrices, $\boldsymbol{B} \triangleq (B_i)_{i=1}^N$ is the sequence of input matrices, $\boldsymbol{C} \triangleq (C_i)_{i=1}^N$ is the sequence of output matrices, while $\boldsymbol{D} \triangleq (D_i)_{i=1}^N$ is the sequence of direct transition matrices. Finally, the initial state and the initial operational mode are x_0 and θ_0, respectively. For ease of notation, from here on, we denote the initial condition as $\phi \triangleq (\mathrm{x}_0, \theta_0)$, and define $\psi_k \triangleq (\mathrm{x}_k, \theta_k)$.

The transition probabilities between the operational modes $i, j \in \mathbb{M}$ of an MJLS are defined as

$$p_{ij}(k) \triangleq \mathrm{Pr}(\theta_{k+1} = j \mid \theta_k = i) \geq 0, \quad \sum_{j=1}^N p_{ij}(k) = 1. \quad (2)$$

Since the probability distribution of a random jump variable θ_k is its probability mass function, $\forall i \in \mathbb{M}$ we have that

$p_i(k) \triangleq \mathrm{Pr}(\theta_k = i)$ and the initial probability distribution is then denoted by a (column) vector $\mathrm{p}_0 \triangleq [p_i(0)]_{i=1}^N$. When the initial operational mode θ_0 is known (to be $\vartheta \in \mathbb{M}$), the information on p_0 may be omitted, considering that $p_\vartheta(0) = 1$ almost surely. Clearly, the distribution p_k of θ_k evolves according to the transition probabilities, i.e.

$$p_j(k+1) = \sum_{i=1}^N p_i(k)p_{ij}(k). \quad (3)$$

A. Polytopic time-inhomogeneous model

In this paper we assume that the transition probabilities are **unknown** and **time-varying** within a bounded set.

Assumption 1: The transition probability matrix (TPM) $P(k) = [p_{ij}(k)]_{i,j=1}^N$ is **polytopic time-inhomogeneous**, i.e.

$$P(k) = \sum_{v=1}^V \lambda_v(k)P_v, \ \lambda_v(k) \geq 0, \ \sum_{v=1}^V \lambda_v(k) = 1, \quad (4)$$

$\forall k \in \mathbb{Z}_0$, where V is a number of vertices of a convex polytope of TPMs $P_v = [p_{ij}^{(v)}]_{i,j=1}^N$, and $\lambda_v(k)$ are unmeasurable.

We denote by $P_{i\bullet}(k)$ the i-th row of $P(k)$, which by Assumption 1 belongs to a polytopic set of stochastic vectors. This notation allows us to denote $P(k)$ as $P_{[1,N]\bullet}(k)$, underlining the fact that in this case the matrix is interpreted row by row. We also slightly abuse our notation by indicating transition probability sequences of length T as $P_{\theta\bullet} \triangleq (P_{\theta_t\bullet}(t))_{t=0}^{T-1}$.

B. Finite-horizon optimal control

Following the line of [10], it is immediate to verify that the solution to the mode-dependent quadratic optimal control problem for PTI MJLSs described above, in finite-horizon case can be obtained from a state-dependent set of coupled difference Riccati equations (CDREs). Specifically, when random variables $\{\mathrm{x}_t, \theta_t\}_{t=0}^k$ are available to the controller and generate a σ-algebra $\mathcal{F}$, so $\mathcal{F}_k \subset \mathcal{F}_{k+1} \subset \mathcal{F}$, the optimal $\mathcal{F}_k$-measurable state-feedback controller $\mathrm{u} \triangleq (\mathrm{u}_t)_{t=0}^{T-1}$ that minimizes the quadratic functional cost associated to the closed loop system over a finite-time horizon, for a worst possible sequence of transition probabilities between the operational modes, is obtained as follows.

Let $\boldsymbol{Z} \triangleq (Z_i)_{i=1}^N$, with $Z_i \in \mathbb{F}_0^{n_x,n_x}$, be a sequence of the terminal cost weighting matrices. Then the optimal cost of robust control for the horizon of length T is defined as

$$\mathcal{J}_T(\phi) \triangleq \min_{\mathrm{u}} \max_{P_{\theta\bullet}} \sum_{k=0}^{T-1} \mathbb{E}(\|\mathrm{z}_k\|^2) + \mathbb{E}(\mathrm{x}_T^* Z_{\theta_T}\mathrm{x}_T). \quad (5)$$

Intuitively, the notation $\mathcal{J}_T(\phi)$ indicates that the the optimal cost is attained in T time steps, by starting from ϕ.

By a standard result of dynamic programming [11], we have that a generic cost at time step k, when there are $T-k$ time steps left to the end of the control time horizon, is

$$J_{T-k}(\psi_k, \mathrm{u}_k, P_{\theta_k\bullet}(k)) = \mathbb{E}(\|\mathrm{z}_k\|^2 + \mathcal{J}_{T-k-1}(\psi_{k+1}) \mid \mathcal{F}_k), \quad (6)$$

where the cost-to-go function is defined as

$$\mathcal{J}_{T-k}(\psi_k) = \min_{\mathrm{u}_k} \max_{P_{\theta_k\bullet}(k)} J_{T-k}(\psi_k, \mathrm{u}_k, P_{\theta_k\bullet}(k)). \quad (7)$$

By the definition of the expected value, we have from (6) that the cost-to-go function (7) is equal to

$$\mathcal{J}_{T-k}(\psi_k) = \min_{\mathrm{u}_k} \max_{P_{[1,N]\bullet}(k)} \sum_{i=1}^N p_i(k)J_{T-k}(\psi_k, \mathrm{u}_k, P_{i\bullet}(k)), \quad (8)$$

where we emphasize the fact that the cost-to-go function is determined by a set of N generic costs, each one associated to a different row of the same TPM $P(k)$.

We assume without loss of generality [1, p.74, Remark 4.1] that for all $i \in \mathbb{M}$ $\quad C_i^* D_i = 0, \quad D_i^* D_i \in \mathbb{F}_0^{n_u, n_u}$. $\quad$ (9) Then via some manipulations described in [10] and explained in detail in [8], we can prove that at each time step k, the maximum in transition probabilities of a generic cost (6) is attained on one of the V vertices of the convex polytope of stochastic matrices P_v that bound the values of the uncertain and time-varying TPM $P(k)$. Thus, to find a worst possible sequence of transition probabilities we need to consider only the TPMs corresponding to the vertices P_v. For each of these vertices, the minimum in u_k of a generic cost (6) is achieved by a state-feedback controller derived from the solution to CRDEs that is obtained by the following backward recursion:

$$R_i^{(v_l)}(k) = \left(D_i^* D_i + B_i^* \sum_{j=1}^N p_{ij}^{(v)} X_j^{(l)}(k+1) B_i \right)^{-1}, \quad (10)$$

$$K_i^{(v_l)}(k) = -R_i^{(v_l)}(k) B_i^* \sum_{j=1}^N p_{ij}^{(v)} X_j^{(l)}(k+1) A_i, \quad (11)$$

$$X_i^{(v_l)}(k) \triangleq C_i^* C_i + A_i^* \sum_{j=1}^N p_{ij}^{(v)} X_j^{(l)}(k+1) A_i + \\ A_i^* \sum_{j=1}^N p_{ij}^{(v)} X_j^{(l)}(k+1) B_i K_i^{(v_l)}(k), \quad (12)$$

where the superscript (v_l) indicates an element obtained with transition probabilities corresponding to vertex $v \leq V$ from the l-th solution to CDRE at the next time step, $l \leq L_{k+1}$. At the last time step we have $L_T = 1$, so that $X_i^{(1)}(T) = Z_i$.

To each $\boldsymbol{X}_{v_l}(k) \triangleq (X_i^{(v_l)}(k))_{i=1}^N$ corresponds a cost

$$\mathcal{J}_{T-k}^{(v_l)}(\psi_k) = x_k^* \left(\sum_{i=1}^N p_i^{(v)}(k) X_i^{(v_l)}(k) \right) x_k, \quad (13)$$

where $p_i^{(v)}(k)$ is obtained by forward recursion starting from the initial probability distribution p_0 via (3), i.e.,

$$p_i^{(v)}(k) = \sum_{j=1}^N p_j(k-1) p_{ji}^{(v)}. \quad (14)$$

Then, for any x_k, the cost-to-go is obtained simply as

$$\mathcal{J}_{T-k}(\psi_k) = \max_{v_l} \mathcal{J}_{T-k}^{(v_l)}(\psi_k). \quad (15)$$

After denoting by $\bar{v}_k = \arg \max_{v_l} \mathcal{J}_{T-k}(\psi_k)$ the index of the solution to CDRE corresponding to the cost-to-go, we can write the optimal control input as

$$u_k = K_{\theta_k}^{(\bar{v}_k)}(k) x_k. \quad (16)$$

Since the terminal cost weighting matrices and the set of vertices of the convex polytope that bounds the values of TPM are known, the solutions to CRDEs $(10)-(12)$ can be computed off-line. The online controller relies on $(13)-(16)$ and the number of solutions to CRDEs to consider may act as a bottleneck, since at $T-k$ time steps left to the end of the control time horizon, there are V^k options to consider.

Nevertheless, some costs (15) obtained from solutions $\boldsymbol{X}_{v_l}(k) \triangleq (X_i^{(v_l)}(k))_{i=1}^N$, $X_i^{(v_l)}(k) \in \mathbb{F}_0^{n_x, n_x}$ may not achieve maximum in transition probabilities and thus will never lead to cost-to-go (7). Such solutions are redundant and could be discarded. So, the set of all VL_{k+1} solutions $\boldsymbol{X}_{v_l}(k)$ can be pruned, obtaining a so-called parsimonious set of $L_k \leq VL_{k+1}$ elements. Specifically, the set $\{\boldsymbol{X}_l(k)\}_{l=1}^{L_k}$ of solutions to CDREs is said to be parsimonious if there is **no other** solution $\boldsymbol{X}_\ell(k)$ such that

$$X_i^{(\ell)}(k) - X_i^{(l)}(k) \in \mathbb{F}_0^{n_x, n_x} \quad \forall i \in \mathbb{M}, l \leq L_k. \quad (17)$$

Note that, if satisfied, (17) means that

$$\sum_{i=1}^N p_i(k) x_k^* \left(X_i^{(\ell)}(k) - X_i^{(l)}(k) \right) x_k \geq 0 \quad \forall x_k, p_k,$$

so the solution $\boldsymbol{X}_l(k)$ would be redundant, to be discarded. We observe that conditions for pruning of redundant solutions, that are similar to (17), were already established for switched Riccati mapping associated with quadratic regulation problem for discrete-time switched linear systems [12]. The number of remaining parsimonious solutions depends on the rate of convergence of the closed-loop system, as will be discussed in the next sections.

C. Infinite-horizon optimal control problem

When the control horizon T goes to infinity, the terminal cost will never be incurred, so the optimal cost of robust control (5) becomes

$$\mathcal{J}_\infty(\phi) \triangleq \min_u \max_{P_{\theta\bullet}} \sum_{k=0}^\infty \mathbb{E}\left(\|z_k\|^2 \right). \quad (18)$$

In this paper, we are interested in the steady-state stabilizing solution to the optimal state-feedback control problem, i.e., to find analytical expression of $u = (u_k)_{k=0}^\infty$ that achieves (18) and stabilizes the MJLS (1) in the mean square sense.

III. MEAN SQUARE STABILIZABILITY

We recall from [1, pp. $36-37$, Definition 3.8, Remark 3.10] that an autonomous MJLS $x_{k+1} = A_{\theta_k} x_k$ is mean square stable (MSS) if for any initial conditions x_0 and θ_0, one has

$$\lim_{k \to \infty} \mathbb{E}(x_k) = 0, \qquad \lim_{k \to \infty} \mathbb{E}(x_k x_k^*) = 0. \quad (19)$$

In [9, Theorems 1 and 3] we have proved that in PTI setting provided by Assumption 1, the system is MSS if and only if the joint spectral radius (see [13] and references therein for an overview) of a finite family of matrices (associated to the second moment of the MJLS) is smaller than 1, and that the mean square stability is equivalent to the exponential mean square stability (EMSS) and to the stochastic stability (SS). Formally, let $_\mathbb{V}\boldsymbol{\Lambda} \triangleq \{\Lambda_v\}_{v=1}^V$ be a set of matrices related to the second moment of x_k, with

$$\Lambda_v \triangleq \left(P_v^T \otimes I_{n_x^2} \right) \left(\bigoplus_{i=1}^N \left(\bar{A}_i \otimes A_i \right) \right). \quad (20)$$

Then, for a PTI MJLS as in (1), (4), the following statements are **equivalent**:

1) $\hat{\rho}(_\mathbb{V}\boldsymbol{\Lambda}) < 1$;
2) the system is MSS, i.e., it satisfies (19) $\forall x_0, \theta_0$;
3) the system is EMSS, i.e., $\forall k \in \mathbb{Z}_0, x_0, \theta_0$, and for some reals $\beta \geq 1$, $0 < \zeta < 1$, one has that
$$\mathbb{E}\left(\|x_k\|^2 \right) \leq \beta \zeta^k \|x_0\|^2; \quad (21)$$
4) the system is SS, i.e. $\forall x_0, \theta_0$, one has that
$$\sum_{k=0}^\infty \mathbb{E}\left(\|x_k\|^2 \right) < \infty. \quad (22)$$

As in [1, p. 57, Definition 3.40], we will say that the pair $(\boldsymbol{A}, \boldsymbol{B})$ of N-sequences of state and control input matrices related to all operational modes of the system (1), is **mean square stabilizable** if there exists a sequences of control matrices $\boldsymbol{K} = (K_i)_{i=1}^N$ such that the system (1) with synchronous state feedback controller $u_k = K_{\theta_k} x_k$ is MSS. In this case, $\boldsymbol{K}$ is said to stabilize the pair $(\boldsymbol{A}, \boldsymbol{B})$.

Since the controller $u_k = K_{\theta_k} x_k$ gives to the system (1) an autonomous form, i.e. $x_{k+1} = (A_{\theta_k} + B_{\theta_k} K_{\theta_k}) x_k = \Gamma_{\theta_k} x_k$, we can apply the previous result on stability equivalence to the controlled system, after substituting in (20) A_i with $\Gamma_i \triangleq (A_i + B_i K_i)$, $\forall i \in \mathbb{M}$. Specifically, let $_\mathbb{V}\boldsymbol{\Delta} \triangleq \{\Delta_v\}_{v=1}^V$, $\Delta_v \triangleq$

$\left(P_v^T \otimes I_{n_x^2}\right)\left(\bigoplus_{i=1}^{N}\left(\bar{\Gamma}_i \otimes \Gamma_i\right)\right)$, be a set of matrices related to the second moment of the controlled state x_k, then the system (1) is stabilizable if and only if there exists a sequence $\boldsymbol{K} = (K_i)_{i=1}^{N}$ of control matrices such that $\hat{\rho}(_{\mathbb{V}}\boldsymbol{\Delta}) < 1$.

We observe that, while it is NP-hard to decide whether $\hat{\rho}(_{\mathbb{V}}\boldsymbol{\Delta}) < 1$ [9, Theorem 2], there exist computationally efficient approximations of the joint spectral radius ([13], [14], [15], [16]), allowing us to verify whether the system is stabilized by a given N-sequence $\boldsymbol{K}$ of control matrices.

For the sake of completeness, we extend the definition of the mean square detectability [1, p. 57, Definition 3.41] to PTI MJLSs by stating that the pair $(\boldsymbol{C}, \boldsymbol{A})$ of N-sequences of output and state matrices related to all operational modes of the system (1) is **mean square detectable** if there exists a sequence of filter gain matrices $\boldsymbol{G} \triangleq (G_i)_{i=1}^{N}$, $G_i \in \mathbb{F}^{n_x, n_z}$, such that for $\boldsymbol{F} \triangleq (F_i)_{i=1}^{N}$, $F_i \triangleq (A_i + G_i C_i)$, $_{\mathbb{V}}\boldsymbol{\Phi} \triangleq \{\Phi_v\}_{v=1}^{V}$, $\Phi_v \triangleq \left(P_v^T \otimes I_{n_x^2}\right)\left(\bigoplus_{i=1}^{N}\left(\bar{F}_i \otimes F_i\right)\right)$, one has that $\hat{\rho}(_{\mathbb{V}}\boldsymbol{\Phi}) < 1$. This definition ensures that the evolution of the observation error for the system (1) with null direct transition matrices and the synchronous full-order Markov jump filter having a structure similar to the structure of the Luenberger observer is mean square stable [8]. We will see in the next section that the mean square detectability (defined explicitly for PTI MJLSs) ensures that the stabilizing solution to a set of coupled algebraic Riccati equations exists and is unique.

As a final remark, we underline that the mean square stabilizability and detectability tests [1, pp. 57–59, Propositions 3.42, 3.43] for time-homogeneous MJLSs can be easily extended to PTI case, with the requirement that the linear matrix inequalities (LMIs) are satisfied for all vertices of the polytopic TPM. Obviously, these tests provide necessary, but not sufficient conditions, so the feasible solutions $\boldsymbol{K}$ and $\boldsymbol{G}$ should be tested also through $\hat{\rho}(_{\mathbb{V}}\boldsymbol{\Delta})$ and $\hat{\rho}(_{\mathbb{V}}\boldsymbol{\Phi})$.

IV. COUPLED ALGEBRAIC RICCATI EQUATIONS

When the MJLS is governed by a Markov chain with a stationary TPM, which is known exactly, if the system is mean square stabilizable and detectable, then there exists the mean square stabilizing solution for the set of coupled algebraic Riccati equations (CAREs) that provides an optimal state-feedback control law and achieves the optimal cost. Furthermore, in this case the solution of CDREs converges to the unique solution of the related CAREs, which coincides with the maximal solution of the same problem and it can be obtained numerically via a certain LMI optimization problem [1, pp. 78–81, 203–228].

In the rest of this paper we generalize the aforementioned result to MJLSs with polytopic time-inhomogeneous transition probabilities, as by Assumption 1.

Definition 1: We say that the $\{\hat{\boldsymbol{X}}_l\}_{l=1}^{L}$, $\hat{\boldsymbol{X}}_l \triangleq (\hat{X}_i^{(l)})_{i=1}^{N}$, $\hat{X}_i^{(l)} \in \mathbb{F}_0^{n_x, n_x}$, $L \leq V$, is the **stabilizing solution** for the **control CAREs** associated to MJLS (1) in PTI setting (4) if $\forall i \in \mathbb{M}$ and $\forall l \leq L$, it satisfies

$$\hat{R}_i^{(l)} = \left(D_i^* D_i + B_i^* \sum\nolimits_{j=1}^{N} p_{ij}^{(l)} \hat{X}_j^{(l)} B_i\right)^{-1}, \quad (23)$$

$$\hat{K}_i^{(l)} = -\hat{R}_i^{(l)} B_i^* \sum\nolimits_{j=1}^{N} p_{ij}^{(l)} \hat{X}_j^{(l)} A_i, \quad (24)$$

$$\hat{X}_i^{(l)} = C_i^* C_i + A_i^* \sum\nolimits_{j=1}^{N} p_{ij}^{(l)} \hat{X}_j^{(l)} A_i + A_i^* \sum\nolimits_{j=1}^{N} p_{ij}^{(l)} \hat{X}_j^{(l)} B_i \hat{K}_i^{(l)}, \quad (25)$$

and $\forall \Gamma_i^{(l)} \triangleq \left(A_i + B_i \hat{K}_i^{(l)}\right)$, $_{\mathbb{V}}\boldsymbol{\Delta}^{(l)} \triangleq \left\{\Delta_v^{(l)}\right\}_{v=1}^{V}$, with

$$\Delta_v^{(l)} \triangleq \left(P_v^T \otimes I_{n_x^2}\right)\left(\bigoplus_{i=1}^{N}\left(\bar{\Gamma}_i^{(l)} \otimes \Gamma_i^{(l)}\right)\right), \quad (26)$$

it satisfies $\hat{\rho}(_{\mathbb{V}}\boldsymbol{\Delta}^{(l)}) < 1$, and if $L < V$, every solution $\hat{\boldsymbol{X}}_\ell$, $L < \ell \leq V$ satisfying (23)–(25) is such that for some $l \leq L$

$$\hat{X}_i^{(l)} - \hat{X}_i^{(\ell)} \in \mathbb{F}_0^{n_x, n_x} \quad \forall i \in \mathbb{M}. \quad (27)$$

In words, Definition 1 states that, when the TPM of an MJLS is unknown and time-varying within a bounded set as by Assumption 1, instead of one set of N coupled algebraic Riccati equations, there are up to V sets of CAREs, each one associated to a different vertex of a convex polytope of transition probabilities. If a solution corresponding to some vertex ℓ satisfies (27), such solution is redundant and should not be considered, because it will never give the optimal cost (18). All the remaining solutions form a parsimonious set, since they produce the smallest set of solutions of CAREs that achieves the optimal cost (18) for any value of the initial state x_0. So the state space is partitioned in $L \leq V$ regions, each one with a different optimal control law. Each controller will stabilize the system if and only if the joint spectral radius of a set of matrices related to the second moment of the controlled system's state is smaller than 1, as reiterated in the central part of the definition. When existing, the defined stabilizing solution for the control CAREs is **unique**, because from (23)–(25) each $\hat{\boldsymbol{X}}_l$ associated to a partition of state-space is a parsimonious solution to a set of CAREs where the transition probabilities are stationary and known, so each $\hat{\boldsymbol{X}}_l$ is unique and can be computed numerically [1, p. 215]:

$$\hat{\boldsymbol{X}}_l = \arg\max_{\boldsymbol{X}_l} \text{tr}\left(\sum\nolimits_{i=1}^{N} X_i^{(l)}\right) \text{ subject to}$$
$$\boldsymbol{X}_l = \left(X_i^{(l)}\right)_{i=1}^{N}, \text{ where } \forall i \in \mathbb{M}, \quad X_i^{(l)} = \left(X_i^{(l)}\right)^T, \text{ and}$$
$$D_i^* D_i + B_i^* \sum\nolimits_{j=1}^{N} p_{ij}^{(l)} X_j^{(l)} B_i \succ 0,$$

$$\begin{bmatrix} -X_i + A_i^* \sum_{j=1}^{N} p_{ij}^{(l)} X_j^{(l)} A_i + C_i^* C_i & A_i^* \sum_{j=1}^{N} p_{ij}^{(l)} X_j^{(l)} B_i \\ B_i^* \sum_{j=1}^{N} p_{ij}^{(l)} X_j^{(l)} A_i & D_i^* D_i + B_i^* \sum_{j=1}^{N} p_{ij}^{(l)} X_j^{(l)} B_i \end{bmatrix} \succeq 0.$$

The LMI optimization problem above can be easily implemented in Matlab-based Robust Control Toolbox [17] and solved via its solver `mincx`, as illustrated in Section V.

It is worth mentioning that if the solution to the above LMI optimization problem exists for an $l \leq V$, then the spectral radius of the matrix $\Delta_l^{(l)}$ associated to the second moment of the MJLS with TPM P_l is less then one, which is a necessary but not sufficient condition for having $\hat{\rho}(_{\mathbb{V}}\boldsymbol{\Delta}^{(l)}) < 1$. Thus, the last condition should be checked separately $\forall \hat{\boldsymbol{K}}_l = (\hat{K}_i^{(l)})_{i=1}^{N}$.

We stress that the mean square detectability of the pair $(\boldsymbol{C}, \boldsymbol{A})$ for a PTI MJLS implies the mean square detectability for a MJLS with stationary TPM P_l (since $\rho(\Phi_v) \leq \hat{\rho}(_{\mathbb{V}}\boldsymbol{\Phi})$), which in turn ensures that the stabilizing solution to a set of CAREs exists if the system is also mean square stabilizable [1, p. 218, Corollary A.16]. By [1, p. 42, Proposition 3.20] this solution is unique.

Let us define the robust control cost function associated to the stabilizing solution for the control CAREs as

$$\hat{\mathcal{J}}_\infty(\psi_k) = \max_l x_k^* \left(\sum\nolimits_{i=1}^{N} p_i^{(l)}(k) \hat{X}_i^{(l)}\right) x_k, \quad (28)$$

with $p_i^{(l)}(k)$ computed via (14), and

$$\hat{v}_k = \arg\max_l \hat{\mathcal{J}}_\infty(\psi_k). \qquad (29)$$

Then we can present the first main result of this paper.

Theorem 1: Suppose that the stabilizing solution $\{\hat{\boldsymbol{X}}_l\}_{l=1}^L$ for the control CAREs associated to MJLS (1) in PTI setting (4) exists. Then the control law $\hat{u} = (\hat{u}_k)_{k=0}^\infty$, where $\hat{u}_k = \hat{K}_{\theta_k}^{(\hat{v}_k)}x_k$, stabilizes the system in the mean square sense and achieves the optimal cost (18), which is given by $\hat{\mathcal{J}}_\infty(\phi)$.

Proof: Since the solution $\{\hat{\boldsymbol{X}}_l\}_{l=1}^L$ of the control CAREs is stabilizing, by Definition 1 it gives us a set $\{\hat{\boldsymbol{K}}_l\}_{l=1}^L$, where $\hat{\boldsymbol{K}}_l = (\hat{K}_i^{(l)})_{i=1}^N$ and $\hat{K}_i^{(l)}$ is provided by (24), such that $\hat{\rho}(_\mathbb{V}\boldsymbol{\Delta}^{(l)}) < 1 \ \forall l$. So, the system (1) with TPM as in (4) and the control law $\hat{u} = (\hat{u}_k)_{k=0}^\infty$, $\hat{u}_k = \hat{K}_{\theta_k}^{(\hat{v}_k)}x_k$, is MSS. Thus, by the stability equivalence, the system is also stochastically stable and $\lim_{k\to\infty} \mathbb{E}(\|x_k\|^2) = 0$, as proved in [9]. Now, for what concerns the optimal cost $\mathcal{J}_\infty(\phi)$, let $\mathcal{U}$ be a set of all mean square stabilizing control laws. For any law $u \in \mathcal{U}$, $u = (u_k)_{k=0}^\infty$, we have from (1), (23)–(25), and the fact that the maximum in transition probabilities of the generic cost at time k is attained on a vertex, denoted by $\hat{v}_k$, of the convex polytope of TPMs, for all x_k and $\hat{X}_{\theta_k}^{(l)}$, with $l \le L \le V$, that

$$\max_{P_{\theta_k\bullet}(k)} \mathbb{E}\Big(x_{k+1}^*\hat{X}_{\theta_{k+1}}^{(l)}x_{k+1} - x_k^*\hat{X}_{\theta_k}^{(l)}x_k + u_k^*D_{\theta_k}^*D_{\theta_k}u_k \mid \mathcal{F}_k\Big) =$$
$$\mathbb{E}\Big(\big(u_k - \hat{K}_{\theta_k}^{(\hat{v}_k)}x_k\big)^*\big(\hat{R}_{\theta_k}^{(\hat{v}_k)}\big)^{-1}\big(u_k - \hat{K}_{\theta_k}^{(\hat{v}_k)}x_k\big) - x_k^*C_{\theta_k}^*C_{\theta_k}x_k\Big),$$

which, together with (1) and (9), implies that

$$\mathbb{E}(\|z_k\|^2) = \mathbb{E}(z_k^*z_k) = \mathbb{E}(x_k^*C_{\theta_k}^*C_{\theta_k}x_k + u_k^*D_{\theta_k}^*D_{\theta_k}u_k) =$$
$$\max_{P_{\theta_k\bullet}(k)}\mathbb{E}\Big(x_k^*\hat{X}_{\theta_k}^{(l)}x_k - x_{k+1}^*\hat{X}_{\theta_{k+1}}^{(l)}x_{k+1} + \Big\|\big(\hat{R}_{\theta_k}^{(\hat{v}_k)}\big)^{-\frac{1}{2}}\big(u_k - \hat{K}_{\theta_k}^{(\hat{v}_k)}x_k\big)\Big\|^2\Big).$$

Since for all $u \in \mathcal{U}$ we have that $\lim_{k\to\infty} \mathbb{E}(\|x_k\|^2) = 0$, and, consequently, $\lim_{k\to\infty} \mathbb{E}(x_k^*\hat{X}_{\theta_k}^{(l)}x_k) = 0$, it follows that

$$\max_{P_{\theta\bullet}}\sum_{k=0}^\infty \mathbb{E}\Big(\Big\|\big(\hat{R}_{\theta_k}^{(\hat{v}_k)}\big)^{-\frac{1}{2}}\big(u_k - \hat{K}_{\theta_k}^{(\hat{v}_k)}x_k\big)\Big\|^2\Big) + \mathbb{E}\big(x_0^*\hat{X}_{\theta_0}^{(l)}x_0\big)$$

is minimized for $u = \hat{u}$. Then, considering also the definition of the expected value and (1), it follows that $\mathcal{J}_\infty(\phi)$ equals to $\mathbb{E}\big(x_0^*\hat{X}_{\theta_0}^{(\hat{v}_0)}x_0\big) = x_0^*\big(\sum_{i=1}^N p_i(0)\hat{X}_i^{(\hat{v}_0)}\big)x_0 = \hat{\mathcal{J}}_\infty(\phi)$. ∎

We observe that, by stability equivalence, the MJLS with control law $(\hat{u}_k)_{k=0}^\infty$, $\hat{u}_k = \hat{K}_{\theta_k}^{(\hat{v}_k)}x_k$, is EMSS, so it generates a trajectory satisfying (21) $\forall k \in \mathbb{Z}_0$, with e.g.

$$\max_l \hat{\rho}(_\mathbb{V}\boldsymbol{\Delta}^{(l)}) \le \zeta < 1 \text{ and } \beta \ge n_x N, \ \forall k \ge k' \ge 0. \qquad (30)$$

See [9, proof of Theorem 3] for additional details.

Now, we show that, when a MJLS is mean square stabilizable, the optimal T-horizon cost function $\mathcal{J}_T(\phi)$ corresponding to the solution to CDREs (12) converges exponentially fast to the optimal infinite-horizon cost function $\hat{\mathcal{J}}_\infty(\phi)$ related to the stabilizing solution to CAREs (25).

The next lemma provides a bound on the optimal cost of finite-horizon robust control, when the MJLS is stabilizible, and is the key in proving the convergence result.

Lemma 1: Suppose that MJLS (1) with PTI TPM (4) is mean square stabilizable and detectable. Then, there exists

$k' \in \mathbb{Z}_0$, such that for any $T \ge k'$, we have that the optimal T-horizon cost $\mathcal{J}_T(\phi)$ of robust control can be bounded as

$$\mathcal{J}_T(\phi) \le \mathbb{E}\Big(\big\|\hat{X}_{\theta_0}^{(\hat{v}_0)}\big\| + \beta\zeta^T\|Z_{\theta_T}\|\Big)\|x_0\|^2, \qquad (31)$$

where $\hat{X}_{\theta_0}^{(\hat{v}_0)}$ is given by (25), $\hat{v}_0$ by (29), β and ζ by (30).

Proof: By hypothesis, the considered MJLS is mean square stabilizable and detectable. So, the stabilizing solution $\{\hat{\boldsymbol{X}}_l\}_{l=1}^V$ for the control CAREs (25) exists and is unique. Thus, following the line of reasoning of [18, Lemma 4], consider the cost $\hat{\mathcal{J}}_T(\phi)$ of robust control related to the trajectory $(\hat{x}_k)_{k=1}^T$ generated from x_0 by application of the infinite-horizon stabilizing optimal control law $(\hat{u}_k)_{k=0}^{T-1}$, with $\hat{u}_k = \hat{K}_{\theta_k}^{(\hat{v}_k)}x_k$ obtained via (23)–(25). By Bellman's principle of optimality, stating that any segment of an optimal trajectory must be the optimal trajectory joining the two end points of the segment, and taking into account Definition 1, (21), (30), linearity of the expected value, sub-multiplicative property of the matrix norm, and that $Z_i \succeq 0$, $\hat{X}_i^{(l)} \succeq 0 \ \forall i \in \mathbb{M}$, $\forall l \le L \le V$, it follows that

$$\hat{\mathcal{J}}_T(\phi) = \hat{\mathcal{J}}_\infty(\phi) - \hat{\mathcal{J}}_\infty(\hat{x}_T, \theta_T) + \mathbb{E}(\hat{x}_T^*Z_{\theta_T}\hat{x}_T)$$
$$= \mathbb{E}\Big(\Big\|\big(\hat{X}_{\theta_0}^{(\hat{v}_0)}\big)^{\frac{1}{2}}x_0\Big\|^2 - \Big\|\big(\hat{X}_{\theta_T}^{(\hat{v}_T)}\big)^{\frac{1}{2}}\hat{x}_T\Big\|^2 + \Big\|\big(Z_{\theta_T}\big)^{\frac{1}{2}}\hat{x}_T\Big\|^2\Big)$$
$$\le \mathbb{E}\Big(\big\|\hat{X}_{\theta_0}^{(\hat{v}_0)}\big\|\|x_0\|^2 + \|Z_{\theta_T}\|\|\hat{x}_T\|^2\Big)$$
$$\le \mathbb{E}\Big(\big\|\hat{X}_{\theta_0}^{(\hat{v}_0)}\big\| + \beta\zeta^T\|Z_{\theta_T}\|\Big)\|x_0\|^2. \qquad (32)$$

Since the minimal T-horizon cost for the worst possible sequence of the transition probabilities is $\mathcal{J}_T(\phi)$, we have that $\mathcal{J}_T(\phi) \le \hat{\mathcal{J}}_T(\phi)$, and by (32) the lemma is proved. ∎

The next theorem shows that the sequence $(\mathcal{J}_T(\phi))_{T=0}^\infty$ of the finite-horizon optimal robust control cost functions converges to the infinite-horizon cost function $\hat{\mathcal{J}}_\infty(\phi)$.

Theorem 2: Suppose that MJLS (1) with PTI TPM (4) is mean square stabilizable and detectable. Then, for any initial condition ϕ, one has that $\lim_{T\to\infty} \mathcal{J}_T(\phi) = \hat{\mathcal{J}}_\infty(\phi)$.

Proof: By hypothesis, the considered MJLS is mean square stabilizable and detectable. So, the stabilizing solution $\{\hat{\boldsymbol{X}}_l\}_{l=1}^V$ for the control CAREs (25) exists and is unique. By Theorem 1, the robust control cost function $\hat{\mathcal{J}}_\infty(\phi)$ associated to $\{\hat{\boldsymbol{X}}_l\}_{l=1}^V$ is the optimal infinite-horizon cost function $\forall\phi$, so $\hat{\mathcal{J}}_\infty(\phi)$ is minimal cost for the worst possible sequence of the transition probabilities. Then, by Lemma 1, there exists k' such that $\mathcal{J}_T(\phi) \le \hat{\mathcal{J}}_T(\phi)$, for all $T \ge k'$. Finally, from (32) and (30), it follows that $\lim_{T\to\infty} \mathcal{J}_T(\phi) = \hat{\mathcal{J}}_\infty(\phi)$. ∎

V. ILLUSTRATIVE EXAMPLE

Consider a simple economic system based on Samuelson's multiplier-accelerator model, which can be described by the following equations [2, p. 8]:
$$\mathcal{C}_t = c\mathcal{Z}_{t-1}, \quad \mathcal{I}_t = w(\mathcal{Z}_{t-1} - \mathcal{Z}_{t-2}), \quad \mathcal{Z}_t = \mathcal{C}_t + \mathcal{I}_t + \mathcal{G}_t,$$
where $\mathcal{C}_t$ is the consumption expenditure, $\mathcal{Z}_t$ is the national income, $\mathcal{I}_t$ is the induced private investment, $\mathcal{G}_t$ is the government expenditure, s is the marginal propensity to save, $1/s$ is the multiplier, $c = 1 - s$ is the marginal propensity to consume, i.e., a slope of the consumption versus income curve, w is the accelerator coefficient, and t is the subscript for the discrete time. As can be seen, the

model is highly aggregated, intended primarily for use as a theoretical tool rather than as a realistic representation of the economy. However, it has been widely used in the literature on MJLSs, see e.g. [1], [2] and references therein. Notably, if one sets the past national incomes as system states, and the government expenditure as control input, i.e., $x_k \triangleq [\mathcal{Z}_{t-2}, \mathcal{Z}_{t-1}]^T$, $u_k \triangleq \mathcal{G}_t$, the above system can be rewritten in the state-space form as in (1). This system has three modes of operation, namely 1, "norm", with s (or w) in mid-range; 2, "boom", having s in low range (or w in high range); 3, "slump", where s is in high range (or w in low range). As a rationale for this terminology, one expects the marginal propensity to save s to decline in "good" times and increase in the "bad", while the acceleration coefficient w would be expected to exhibit opposite tendencies [19].

The parameters for each of these modes of operation and stochastic matrices used to bound the TPM are borrowed from [1, pp. 180 – 181, Example 8.3]:

$$A_1 = \begin{bmatrix} 0 & 1 \\ -2.2308 & 2.5462 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ -38.9103 & 2.5462 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 1 \\ 4.6384 & -4.7455 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, B_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1.5049 & -1.0709 \\ -1.0709 & 1.6160 \\ 0 & 0 \end{bmatrix}, C_2 = \begin{bmatrix} 10.2036 & -10.3952 \\ -10.3952 & 11.2819 \\ 0 & 0 \end{bmatrix},$$

$$C_3 = \begin{bmatrix} 1.7335 & -1.2255 \\ -1.2255 & 1.6639 \\ 0 & 0 \end{bmatrix}, D_1 = \begin{bmatrix} 0 \\ 0 \\ 1.6125 \end{bmatrix}, D_2 = \begin{bmatrix} 0 \\ 0 \\ 1.0794 \end{bmatrix},$$

$$D_3 = \begin{bmatrix} 0 \\ 0 \\ 1.0540 \end{bmatrix}, P_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, P_3 = \begin{bmatrix} 0.50 & 0.25 & 0.25 \\ 0.20 & 0.50 & 0.30 \\ 0.30 & 0.30 & 0.40 \end{bmatrix},$$

$$P_2 = \begin{bmatrix} 0.83 & 0.09 & 0.08 \\ 0.46 & 0.39 & 0.15 \\ 0.42 & 0.02 & 0.56 \end{bmatrix}, P_1 = \begin{bmatrix} 0.51 & 0.25 & 0.24 \\ 0.14 & 0.55 & 0.31 \\ 0.10 & 0.18 & 0.72 \end{bmatrix}.$$

An approximate value of the TPM is obtained from the historical data of the United States Department of Commerce [2, pp. 8 – 9], while the considered polytopic bounds were proposed by Costa et al. [1, pp. 180 – 181, Example 8.3].

Let us consider the MJLS with TPM $P(k) \in \text{conv}\{P_i\}_{i=1}^4$. This MJLS is unstable, since the spectral radii associated to the matrices Λ_i computed via (20) are $\rho(\Lambda_1) = 31.652$, $\rho(\Lambda_2) = 20.110$, $\rho(\Lambda_3) = 29.962$, and $\rho(\Lambda_4) = 38.910$, and the joint spectral radius $\hat{\rho}(\mathbb{v}\boldsymbol{\Lambda}) \geq \max \rho(\Lambda_v)$.

This system is mean square stabilizable and detectable, so the optimal infinite-horizon robust controller exists. Specifically, there are two possible mode-dependent state-feedback gains, computed from (24) via LMI optimization, and taking into account (27), which are the following:

$$\hat{\boldsymbol{K}}_1 = \begin{pmatrix} \hat{K}_1^{(1)} = [-2.223, 2.400], \\ \hat{K}_2^{(1)} = [-38.860, 2.345], \\ \hat{K}_3^{(1)} = [4.632, -4.890] \end{pmatrix}, \hat{\boldsymbol{K}}_2 = \begin{pmatrix} \hat{K}_1^{(2)} = [-1.921, 1.538], \\ \hat{K}_2^{(2)} = [-38.889, 2.392], \\ \hat{K}_3^{(2)} = [4.511, -5.407] \end{pmatrix}.$$

Notably, the control gain $\hat{\boldsymbol{K}}_1$ is obtained in correspondence of the vertex P_3, while $\hat{\boldsymbol{K}}_2$ is related to the vertex P_4. Both these state-feedback controllers are stabilizing, since

$$\hat{\rho}(\mathbb{v}\hat{\boldsymbol{\Delta}}^{(1)}) < 0.05077, \ \hat{\rho}(\mathbb{v}\hat{\boldsymbol{\Delta}}^{(2)}) < 0.66739,$$

where the value of the joint spectral radius has been computed via the JSR toolbox [16].

Noticeably, since the TPM P_4 treats all operational modes independently, the related control gains $\hat{\boldsymbol{K}}_2$ coincide with those obtained by solving the classical discrete-time algebraic Riccati equations for linear time-independent systems.

The actual choice of the control gain depends on the initial condition ϕ, as by (29). For $\text{x}_0 = [1, 1]^T = \boldsymbol{x}_0$, and $\theta_0 = 1$, we have that the optimal cost $\mathcal{J}_\infty(\boldsymbol{x}_0, 1) = \max\{495.715, 6.161\}$ of the robust control is obtained from the first parsimonious solution to the CAREs. For $\theta_0 = 2$, we have instead that the optimal cost $\mathcal{J}_\infty(\boldsymbol{x}_0, 2) = \max\{2519.877, 3478.062\}$ is achieved with the second solution, while for $\theta_0 = 3$, we find that $\mathcal{J}_\infty(\boldsymbol{x}_0, 3) = \max\{591.376, 3.062\}$, so the control gain to apply is $\hat{K}_3^{(1)}$.

Since the values of the joint spectral radius are relatively small, we have a fast convergence of the finite-horizon solution to steady-state. Let the terminal cost weighting matrices be $Z_1 = 2I_2$, $Z_2 = I_2$, and $Z_3 = 4I_2$. Then, for time horizon $T = 8$ we have already almost the same values of the state-feedback gains, precisely

$$\boldsymbol{K}_1 = \begin{pmatrix} K_1^{(1)} = [-2.223, 2.399], \\ K_2^{(1)} = [-38.860, 2.344], \\ K_3^{(1)} = [4.632, -4.891] \end{pmatrix}, \boldsymbol{K}_2 = \begin{pmatrix} K_1^{(2)} = [-1.921, 1.538], \\ K_2^{(2)} = [-38.889, 2.392], \\ K_3^{(2)} = [4.512, -5.403] \end{pmatrix},$$

while the optimal costs for the different initial operational modes are respectively $\mathcal{J}_8(\boldsymbol{x}_0, 1) = \max\{495.698, 6.160\}$, $\mathcal{J}_8(\boldsymbol{x}_0, 2) = \max\{2519.876, 3478.062\}$, and $\mathcal{J}_8(\boldsymbol{x}_0, 3) = \max\{591.344, 3.212\}$. So, it is evident that for each initial operational mode the values of the finite-horizon optimal costs of the robust control are very close to the values of the optimal costs of the infinite-horizon setting.

It is also worth noting that for any length T of the finite-time horizon with the (integer) values in between 4 and 1000, the maximum number of elements in the parsimonious set of CDREs is obtained for the time step $T-4$ and corresponds to $VL_{T-3} = 4 \cdot 4 = 16 \ll 4^T$.

If we consider the same problem for the MJLS with TPM $P(k) \in \text{conv}\{P_i\}_{i=1}^3$, the illustrated approach gives other two stabilizing control gains, first one associated to the vertex P_1, and the second one, as before, related to the vertex P_3.

$$\tilde{\boldsymbol{K}}_1 = \begin{pmatrix} \tilde{K}_1^{(1)} = [-2.222, 2.393], \\ \tilde{K}_2^{(1)} = [-38.860, 2.331], \\ \tilde{K}_3^{(1)} = [4.629, -4.880] \end{pmatrix}, \tilde{\boldsymbol{K}}_2 = \begin{pmatrix} \tilde{K}_1^{(2)} = [-2.223, 2.400], \\ \tilde{K}_2^{(2)} = [-38.860, 2.345], \\ \tilde{K}_3^{(2)} = [4.632, -4.890] \end{pmatrix},$$

with respective $\hat{\rho}(\mathbb{v}\tilde{\boldsymbol{\Delta}}^{(1)}) < 0.03569$, $\hat{\rho}(\mathbb{v}\tilde{\boldsymbol{\Delta}}^{(2)}) < 0.03610$. The values of the joint spectral radius are particularly small in this setting, so we expect a very fast convergence of the finite-horizon solution to steady-state.

In fact, the optimal cost of the infinite-horizon robust state-feedback control for the initial state $\boldsymbol{x}_0$ and different initial operational modes are $\mathcal{J}_\infty(\boldsymbol{x}_0, 1) = \max\{495.036, 495.715\}$, $\mathcal{J}_\infty(\boldsymbol{x}_0, 2) = \max\{2613.443, 2519.877\}$, and $\mathcal{J}_\infty(\boldsymbol{x}_0, 3) = \max\{366.066, 591.376\}$, while for a finite-time horizon, with the length as short as $T = 5$, we have already exactly the same as for $\tilde{\boldsymbol{K}}_1$ and $\tilde{\boldsymbol{K}}_2$ values of the mode-dependent state-feedback gains (with precision up to the fourth decimal place). The related optimal costs of the

robust control are $\mathcal{J}_5(\boldsymbol{x}_0, 1) = \max\{495.021, 495.715\}$, $\mathcal{J}_5(\boldsymbol{x}_0, 2) = \max\{2613.416, 2519.853\}$, and $\mathcal{J}_5(\boldsymbol{x}_0, 3) = \max\{366.051, 591.358\}$. For $T = 6$ the finite- and infinite-horizon optimal costs coincide (with the same precision, up to the fourth decimal place).

Noticeably, for any length T of the finite-time horizon with the (integer) values in between 4 and 1000, the maximum number of elements in the parsimonious set of CDREs is now obtained for the time step $T-2$ and corresponds to $VL_{T-1} = 3 \cdot 2 = 6 \ll 3^T$.

VI. CONCLUSIONS

In this paper, we generalized the infinite-horizon linear quadratic regulation results of stationary MJLSs to politopic time-inhomogeneous setting, providing an analytical solution that can be computed efficiently. The natural extension of this work is to consider different types of additive process and observation noise, together with partial and delayed information on operational modes.

REFERENCES

[1] O. L. V. Costa, M. D. Fragoso, and R. P. Marques, *Discrete-time Markov jump linear systems*. Springer, 2005.

[2] L. Zhang, T. Yang, P. Shi, and Y. Zhu, *Analysis and design of Markov jump systems with complex transition probabilities*, ser. Stud. Syst. Decis. Control. Springer, 2016, vol. 54.

[3] Y. Long and G.-H. Yang, "Fault detection for a class of nonhomogeneous Markov jump systems based on delta operator approach," *Proc. Inst. Mech. Eng. I*, vol. 227, no. 1, pp. 129–141, 2013.

[4] J. P. Hespanha, P. Naghshtabrizi, and Y. Xu, "A survey of recent results in networked control systems," *Proc. IEEE*, vol. 95, no. 1, pp. 138–162, Jan. 2007.

[5] L. Schenato, B. Sinopoli, M. Franceschetti, K. Poolla, and S. S. Sastry, "Foundations of control and estimation over lossy networks," *Proc. IEEE*, vol. 95, no. 1, pp. 163–187, Jan. 2007.

[6] M. Pajic, S. Sundaram, G. J. Pappas, and R. Mangharam, "The wireless control network: a new approach for control over networks," *IEEE Trans. Autom. Control*, vol. 56, no. 10, pp. 2305–2318, Oct. 2011.

[7] R. Alur, A. D'Innocenzo, K. H. Johansson, G. J. Pappas, and G. Weiss, "Compositional modeling and analysis of multi-hop control networks," *IEEE Trans. Autom. Control*, vol. 56, no. 10, pp. 2345–2357, 2011.

[8] Y. Zacchia Lun, "Stability and optimal control of polytopic time-inhomogeneous Markov jump linear systems," Ph.D. dissertation, Gran Sasso Science Institute (GSSI), L'Aquila, Italy, 2017.

[9] Y. Zacchia Lun, A. D'Innocenzo, and M. D. Di Benedetto, "On stability of time-inhomogeneous Markov jump linear systems," in *55th Conf. Decision Control (CDC)*. IEEE, Dec. 2016, pp. 5527–5532.

[10] Y. Zacchia Lun, A. D'Innocenzo, A. Abate, and M. D. Di Benedetto, "Optimal robust control and a separation principle for polytopic time-inhomogeneous Markov jump linear systems," in *56th Conf. Decision Control (CDC)*. IEEE, Dec. 2017, pp. 6525–6530.

[11] D. P. Bertsekas, *Dynamic programming and optimal control*. Athena Scientific, 1995, vol. I & II.

[12] W. Zhang, J. Hu, and A. Abate, "Infinite-horizon switched LQR problems in discrete time: a suboptimal algorithm with performance analysis," *IEEE Trans. Autom. Control*, vol. 57, no. 7, pp. 1815–1821, 2012.

[13] R. M. Jungers, *The joint spectral radius: theory and applications*, ser. Lecture Notes Control Inform. Sci. Springer, 2009, vol. 385.

[14] V. D. Blondel and Y. Nesterov, "Computationally efficient approximations of the joint spectral radius," *SIAM J. Matrix Anal. Appl.*, vol. 27, no. 1, pp. 256–272, 2005.

[15] V. Y. Protasov, R. M. Jungers, and V. D. Blondel, "Joint spectral characteristics of matrices: a conic programming approach," *SIAM J. Matrix Anal. Appl.*, vol. 31, no. 4, pp. 2146–2162, 2010.

[16] G. Vankeerberghen, J. Hendrickx, and R. M. Jungers, "JSR: A toolbox to compute the joint spectral radius," in *17th Int. Conf. Hybrid Syst.: Comput. Control (HSCC)*. ACM, 2014, pp. 151–156.

[17] G. Balas, R. Chiang, A. Packard, and M. Safonov, "Robust Control Toolbox™User's Guide, version 6.4," *MathWorks® Matlab®*, 2017.

[18] W. Zhang, J. Hu, and A. Abate, "On the value functions of the discrete-time switched LQR problem," *IEEE Trans. Autom. Control*, vol. 54, no. 11, pp. 2669–2674, 2009.

[19] W. P. Blair Jr. and D. D. Sworder, "Feedback control of a class of linear discrete systems with jump parameters and quadratic cost criteria," *Int. J. Control*, vol. 21, no. 5, pp. 833–841, 1975.