Logic and Proof Hilary 2024

Exercise Sheet 2

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Answers to questions marked with $*$ will be posted on the course website.

1*. Using resolution, show that $P_1 \wedge P_2 \wedge P_3$ is a consequence of

$$F := (\neg P_1 \vee P_2) \wedge (\neg P_2 \vee P_3) \wedge (P_1 \vee \neg P_3) \wedge (P_1 \vee P_2 \vee P_3) \,.$$

2*. Simulate a run of the DPLL algorithm with clause learning on the following formula. Use the decision strategy *assign true to the currently unassigned variable of least index.*

$$\underbrace{(\neg p_1 \vee \neg p_2)}_{C_1} \wedge \underbrace{(\neg p_1 \vee p_3)}_{C_2} \wedge \underbrace{(\neg p_3 \vee \neg p_4)}_{C_3} \wedge \underbrace{(p_2 \vee p_4 \vee p_5)}_{C_4} \wedge \underbrace{(\neg p_5 \vee \neg p_6 \vee \neg p_7)}_{C_5}$$
$$\wedge \underbrace{(\neg p_6 \vee p_7 \vee \neg p_8)}_{C_6} \wedge \underbrace{(p_8 \vee \neg p_9)}_{C_7} \wedge \underbrace{(p_8 \vee p_9 \vee \neg p_1)}_{C_8}$$

3*. As suggested in Footnote 1 of Lecture 7, give a formal proof that $C'_1, \ldots, C'_{m'}$ is a pseudo-refutation of PHP_{n-1}.

4. A *renamable Horn formula* is a CNF formula that can be turned into a Horn formula by negating (all occurrences of) some of its variables. For example,

$$(P_1 \vee \neg P_2 \vee \neg P_3) \wedge (P_2 \vee P_3) \wedge (\neg P_1)$$

can be turned into a Horn formula by negating P_1 and P_2.

Given a CNF-formula F, show how to derive a 2-CNF formula G such that G is satisfiable if and only if F is a renamable Horn formula. Show moreover that one can derive a renaming that turns F into a Horn formula from a satisfying assignment for G.

5. Let F be a Horn-CNF formula, with n variables, in which all clauses contains at least two literals. Argue that when run on F, the Walk-SAT algorithm finds a satisfying assignment for F with probabilty at least $1/2$ after $2n^2$ steps.

6. Using resolution, or otherwise, show that there is a polynomial-time algorithm to decide satisfiability of those CNF formulas F in which each propositional variable occurs at most twice. Justify your answer.

(**Hint:** Show how to eliminate a variable from the formula without affecting satisfiability or increasing the number of clauses.)

7. Argue that if the DPLL algorithm is run on a 2-CNF formula then every clause that is learned is either empty or a singleton.

(**Hint:** Compare the DPLL algorithm with the algorithm for 2-SAT in Figure 2, Lecture 4.)

8. *Positive resolution* is a restriction of ordinary resolution, which is defined as follows: derive a resolvent from C_1 and C_2 only if C_1 is a positive clause, i.e., it consists only of positive literals. Show that if F is an unsatisfiable CNF formula then one can derive the empty clause from F using only positive resolution.

9. Suppose that $\boldsymbol{S} \models F$ for some formula F and set of formulas $\boldsymbol{S}$. Show that there is a finite set $\boldsymbol{S}_0 \subseteq \boldsymbol{S}$ such that $\boldsymbol{S}_0 \models F$.

10. Given an undirected graph $G = (V, E)$, a set of vertices $S \subseteq V$ is a *clique* if every pair of distinct vertices $u, v \in S$ is connected by an edge and S is an *independent set* if no pair of distinct vertices $u, v \in S$ is connected by an edge. Now consider the following two statements:

    (A) Every infinite graph either has an infinite clique or an infinite independent set.

    (B) For all k there exists n such that any graph with n vertices has a clique of size k or an independent set of size k.

    The goal of this question is to show that (A) implies (B).[1]

    (a) Carefully formulate the negation of (B).

    (b) Assuming the negation of (B), use the Compactness Theorem to prove the negation of (A), i.e., that there is an infinite graph with no infinite clique and no infinite independent set.

[1] As an optional exercise, beyond the scope of the course, you can try to prove (A). This result is Theorem 1 in `https://www.dpmms.cam.ac.uk/~par31/notes/ramsey.pdf`. Combining this with 7(b) we obtain a proof of (B).